DRAFTSMAN'S MATHEMATICAL MANUAL

INCLUDING PRINCIPLES OF ALGEBRA, USE OF EQUATIONS, AND A SELECTED COLLECTION OF ENGINEERING AND DRAFTING-ROOM PROBLEMS WITH METHODS OF ANALYSIS AND SOLUTION, FOR THE USE OF DRAFTSMEN AND OTHERS REQUIRING THE AID OF PRACTICAL EXAMPLES AS A GUIDE IN SOLVING SIMILAR PROBLEMS

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SECOND EDITION

FIRST PRINTING

NEW YORK

THE INDUSTRIAL PRESS

LONDON: THE MACHINERY PUBLISHING CO., LTD.

Printed in U. S. A.

PREFACE

This book consists chiefly of a selected collection of mathematical engineering and drafting-room problems. The value of practical examples as a means of illustrating the application of mathematical principles is recognized by the authors of all text-books dealing with mathematics, and since the object of examples is to show just how problems of different kinds are analyzed and how mathematical principles are applied in practice, it is evident that much depends upon the examples selected. The examples in this book were submitted originally to the publishers for solution, generally by draftsmen or designers, and they represent the kinds of problems which are of particular interest and importance to those engaged in machine and tool design.

Rare problems involving extremely complex solutions have been excluded, although the solutions of many of the examples given have proved to be more or less difficult for many draftsmen. An effort has been made in each case to present the simplest solutions possible and the ones most readily understood, although more than one method has been included frequently in order to illustrate different ways of obtaining the same final result. The methods of analyzing and solving the numerous examples found in this book can be applied to a great variety of other problems based upon the same fundamental principles. As a glance at the different chapters will show, problems of the same general type have been grouped together as far as possible. This method of classification and arrangement, in conjunction with the regular index, will make it possible to locate whatever problems and solutions either duplicate or closely resemble any given problem.

Credit for the mathematical work which has made it possible to present such a variety of problems, is due to a large number of contributors to Machinery, and notably to J. J. Clark and W. W. Johnson, whose mathematical contributions have appeared frequently in connection with Machinery's "Questions and Answers" page. It is believed that this collection of practical and applied problems will prove of the greatest value, not only as a supplement to existing text-books on mathematics, but as a companion book to standard engineering handbooks.

The Editor.

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CHAPTER I

PRINCIPLES OF ALGEBRA

ENGINEERING practice is based largely upon a knowledge of mathematics. The more exact and complete that knowledge, the less uncertain engineering results become, although too implicit reliance upon the conclusions of purely mathematical investigation may prove to be a serious fault. While mathematical processes are essentially methods of precision, it is evident that they give trustworthy results only when the basic assumptions from which they proceed are correct. Purely mathematical theories serve a useful purpose in engineering, but the work of the draftsman and engineer involves producing results which shall be of practical service. The experienced designer or engineer recognizes that any degree of accuracy that is beyond practical requirements is useless and wasteful of time and energy. Examples of typical calculations from engineering practice serve an important purpose because they demonstrate how mathematical principles are applied under actual working conditions and illustrate the extent to which precise methods are utilized in practice. This book consists largely of practical problems and methods of solution, although the first few chapters review briefly the important principles of algebra and explain the application of logarithms, for the benefit of those who may desire to review these important branches of mathematical work.

Positive and Negative Quantities. The graduations above and below the zero mark on a thermometer scale illustrate the distinction between positive and negative quantities. The degrees on the scale extending upward from the zero point might be called *positive* and be preceded by a plus sign, so that, for instance, $+5$ degrees would mean 5 degrees above zero. The degrees below zero might be called *negative* and be preceded by a minus sign, so that -5 degrees would mean 5 degrees below zero. Ordinary numbers may

also be considered positive and negative in the same way as the graduations on a thermometer scale. When we count 1, 2, 3, etc., we refer to the numbers that are larger than 0 (corresponding to the degrees *above* the zero point), and these numbers are called positive numbers. We can conceive, however, of numbers extending in the opposite direction relative to 0; numbers that are, in fact, less than 0 (corresponding to the degrees below the zero point on the thermometer scale). As these numbers must be expressed by the same figures as the positive numbers, they are designated by a minus sign placed before them. For example, -3 means a number that is as much less than, or beyond 0 in the negative direction as 3 (or, as it might be written, $+3$) is larger than 0 in the positive direction.

A negative value should always be enclosed within parentheses whenever it is written in line with other numbers; for example:

$$17 + (-13) - 3 \times (-0.76)$$

In this example -13 and -0.76 are negative numbers, and by enclosing the whole number, minus sign and all, in parentheses, it is shown that the minus sign is part of the number itself, indicating its negative value. It must be understood that when we say $7 - 4$, then 4 is not a negative number, although it is preceded by a minus sign. In this case the minus sign is simply the sign of subtraction, indicating that 4 is to be subtracted from 7; but 4 is still a positive number or a number that is larger than 0.

It now being clearly understood that positive numbers are all ordinary numbers greater than 0, while negative numbers are conceived of as less than 0 and preceded by a minus sign which is a part of the number itself, the following rules can be given for calculations with negative numbers. The application of the expressions positive and negative to algebraic terms involving symbols instead of figures follows the same rules, as the examples indicate.

Addition of Positive and Negative Numbers. *Rule: A negative number can be added to a positive number by subtracting its numerical value from the positive number.*

Examples:

$$4 + (-3) = 4 - 3 = 1$$
$$16 + (-7) + (-6) = 16 - 7 - 6 = 3$$
$$327 + (-0.5) - 212 = 327 - 0.5 - 212 = 114.5$$

In the last example 212 is not a negative number, because there are no parentheses indicating that the minus sign is a part of the number itself. The minus sign, then, indicates only that 212 is to be subtracted in the ordinary manner.

$$A + (-B) = A - B$$
$$A + (-B - C) = A - B - C$$

The last example shows the application of an important rule relating to the use of parenthesis in algebra. *If a parenthesis is preceded by a + sign, it may be removed, if the terms inside the parentheses retain their signs.*

Subtraction of Positive and Negative Numbers. *A negative number can be subtracted from a positive number by adding its numerical value to the positive number.*

Examples:

$$4 - (-3) = 4 + 3 = 7$$
$$16 - (-7) = 16 + 7 = 23$$
$$327 - (-0.5) - 212 = 327 + 0.5 - 212 = 115.5$$

In the last example, note that 212 is subtracted, because the minus sign in front of it does not indicate that 212 is a negative number.

$$A - (-B) = A + B$$
$$A - (-B - C) = A + B + C$$

Parenthesis Preceded by Minus Sign. The two examples just given are an application of the rule that a *parenthesis preceded by a − sign may be removed if the signs preceding each of the terms inside the parentheses are changed* (+ changed to −, and − to +). Multiplication and division signs are not affected.

Examples:

$$A - (-B + C \div D) = A + B - C \div D$$
$$A - (B - C + D \times E) = A - B + C - D \times E$$

Remember that B in the last example, with no sign in front of it inside of the parentheses, is considered as preceded by a + sign. Hence this + sign is changed to − when removing the parentheses.

As an illustration of the method used when subtracting a negative number from a positive one, assume that we are required to find how many degrees difference there is between 37 degrees above zero and 24 degrees below; this latter may be written (− 24). The difference between the two numbers of degrees mentioned is then:

$$37 - (-24) = 37 + 24 = 61$$

A little thought makes it obvious that this result is right, and the example shows that the rule given is based on correct reasoning.

Positive Number Multiplied or Divided by Negative Number. *Rule: When a positive number is multiplied or divided by a negative number, multiply or divide the numerical values as usual; but the*

product or quotient, respectively, becomes negative. The same rule holds true if a negative number is multiplied or divided by a positive number.

Examples:

$$4 \times (-3) = -12 \qquad (-3) \times 4 = -12$$

$$\frac{15}{-3} = -5 \qquad \frac{-15}{3} = -5$$

$$A \times (-B) = -AB$$

$$\frac{A}{-B} = -\frac{A}{B} \qquad \frac{-A}{B} = -\frac{A}{B}$$

Multiplication and Division of Negative Numbers. *Rule: When two negative numbers are multiplied by each other, the product is positive. When a negative number is divided by another negative number the quotient is positive.*

Examples:

$$(-4) \times (-3) = 12 \qquad \frac{-4}{-3} = 1.333$$

$$(-A) \times (-B) = AB \qquad \frac{-A}{-B} = \frac{A}{B}$$

If, in a subtraction, the number to be subtracted is larger than the number from which it is to be subtracted, the calculation can be carried out by subtracting the smaller number from the larger, and indicating that the remainder is negative.

Examples:

$$3 - 5 = -(5 - 3) = -2$$

In this example 5 cannot, of course, be subtracted from 3, but the numbers are reversed, 3 being subtracted from 5, and the remainder indicated as being negative by placing a minus sign before it.

$$227 - 375 = -(375 - 227) = -148$$

$$7a - 9a = -2a$$

The rules given are highly important in all algebraic calculations, and should be committed to memory.

General Rules for Addition and Subtraction in Algebra. Only *like terms* can be added or subtracted. By like terms are meant those which differ only in their numerical coefficients. For example, $9xy$ and $\frac{1}{2}xy$ are like terms; but $3x$ and $3x^2$ are unlike terms, because here the terms differ with respect to exponents.

$7a^2$, $6a^2$, $\frac{3}{4}a^2$ and a^2 are all like terms. $5a^2$, $5a$, $5ab$, $5b$, and b^2 are all unlike terms.

Unlike terms cannot be added or subtracted, but the addition or

subtraction may be *indicated* by placing + or − signs, respectively, between the terms.

Examples of adding and subtracting like terms are given below:

1.	$a + a = 2a$	7.	$3a + 6a + 2a = 11a$
2.	$x - x = 0$	8.	$5x + 9x + x = 15x$
3.	$2a + a = 3a$	9.	$3xy + 5xy = 8xy$
4.	$2x - x = x$	10.	$ax + 5ax = 6ax$
5.	$3a + 2a = 5a$	11.	$19abc - abc = 18abc$
6.	$5x - 2x = 3x$	12.	$2xyz + xyz = 3xyz$

Examples of addition or subtraction of unlike terms, which cannot be added directly, are given below. Add all like terms where possible.

1. $2xy + xy + a = 3xy + a$
2. $3a + b + 2a = 5a + b$
3. $2a + a - b = 3a - b$
4. $5x - y + 2x = 7x - y$
5. $3xy + 3x + 3y + 2xy + 2x = 5xy + 5x + 3y$

If any terms are enclosed in parentheses, these are removed, the signs of the quantities within the parentheses being changed if the parentheses are preceded by a − sign, according to the rule previously given.

Examples:

1. $a + (b - c) = a + b - c$
2. $a - (b - c) = a - b + c$
3. $2a - (a + b) = 2a - a - b = a - b$
4. $5a + (3a + b) = 5a + 3a + b = 8a + b$
5. $3xy - (5xy - 2y) = 3xy - 5xy + 2y = -2xy + 2y$

The general rules for addition and subtraction of algebraic quantities may be stated as follows:

First remove all parentheses, changing the signs if required by the rule previously given.

Add the coefficients of all positive like terms.

Next, add the coefficients of all negative like terms.

Subtract the less of these two sums from the greater, and prefix the sign of the greater sum to the result; then annex the like symbols to this coefficient.

Examples:

1. $3xy - 5xy + 3xy + 2xy - 2xy = 8xy - 7xy = xy$
2. $2ab - (5ab - 6ab + ab) = 2ab - 5ab + 6ab - ab = 8ab - 6ab = 2ab$

3. $-2ax + 5ax - 3ax + 1 = 5ax - 5ax + 1 = 1$
4. $5a + 6b + 2a - 3b - 2c = 7a + 3b - 2c$
5. $6x - (5y - x) + 2y - (x + 5y) =$
   $6x - 5y + x + 2y - x - 5y = 6x - 8y$
6. $6x - [y - (7x - 4) + (x - y)] =$
   $6x - y + (7x - 4) - (x - y) =$
   $6x - y + 7x - 4 - x + y = 12x - 4$
7. $-(2xy - 2x - 2y) = -2xy + 2x + 2y$, or as it is generally written, $2x + 2y - 2xy$
8. $-[-x - (-y + x)] = +x + (-y + x) = x - y + x$
   $= 2x - y$

Rules for Multiplication in Algebra. The first rule for multiplication in algebra is as follows: *To find the product of two or more quantities, multiply together their coefficients, and prefix this product to the quantities expressed by symbols.* Remember that expressions such as $5ab$ imply the multiplication $5 \times a \times b$.

Examples:

1. $5x \times 3y = 5 \times x \times 3 \times y = 15 \times x \times y = 15xy$
2. $3a \times 4b \times 5c = 3 \times 4 \times 5 \times a \times b \times c = 60abc$
3. $6ab \times 2c = 12abc$
4. $xy \times 3z = 3xyz$

The second rule for multiplication in algebra is: *The sum of the exponents of the letters in the factors to be multiplied, equals the exponent of the same letters in the product.* Remember that a has, in fact, the exponent (^{1}), although we never write it a^1, but simply drop the exponent when it is (^{1}).

Examples:

1. $a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^5$
2. $a \times a^2 = a^3$
3. $a^2b \times a = a^2ab = a^3b$
4. $a^2b \times ab^2 = a^{2+1}\,b^{1+2} = a^3b^3$
5. $a \times a^2b \times ab^2 \times b^3 = a^4b^6$
6. $6ab^2 \times 3a^2b^2 = 18a^3b^4$

The last example illustrates the simple manner in which we may express the two rules given: *The coefficients are multiplied, and the exponents are added together.*

Referring to the rules given for positive and negative numbers, we may formulate the third rule for multiplication as follows: *Two positive* (+) *factors give a positive* (+) *product; two negative* (−) *factors give a positive* (+) *product; but one positive* (+) *and*

one negative (−) factor give a negative (−) product. This rule is often stated more briefly as follows: *Like signs produce plus, and unlike signs produce minus in the product.*

Examples:

1. $2xyz \times 4x^2 = 8x^3yz$
2. $-3ab \times -6ab = +18a^2b^2$
3. $3ab \times -6ab = -18a^2b^2$
4. $-3ab \times 6ab = -18a^2b^2$

Multiplication of Positive and Negative Quantities. When three or more quantities that are not all positive are to be multiplied, the safest method is to multiply two factors at a time, until all have been multiplied. In this way errors in the sign of the product are avoided.

Examples:

1. $2ab \times -3b \times -4a = -6ab^2 \times -4a = +24a^2b^2$
2. $-a \times -b \times -c \times -d = ab \times -c \times -d = -abc \times -d = +abcd$
3. $16a^2b^2 \times -16a^2b^2 \times -a = -256a^4b^4 \times -a = 256a^5b^4$

Expressions within parentheses may be considered as single letters, especially if they are higher than the first power.

4. $(a-b)^2 \times (a-b) = (a-b)^3$
5. $(a+b) \times -(a+b)^2 = -(a+b)^3$

Terms Enclosed in Parentheses. If an expression consisting of several terms is enclosed in parentheses, and this expression is to be multiplied by another term, the multiplication sign is often omitted, the same as between individual letters. For example, $(x-y+z)\,a = (x-y+z) \times a$.

When one factor consists of several terms enclosed in parentheses, and the other factor is a single term, *multiply each of the terms in the parentheses by the single term.* Remember that in this case also like signs produce plus and unlike signs minus.

Examples:

1. $(x+y-t)\,a = ax + ay - at$
2. $b\,(ab - ab^2 + a^2) = ab^2 - ab^3 + a^2b$
3. $-ab\,(a^2 - 2ab + b^2) = -a^3b + 2a^2b^2 - ab^3$
4. $4x\,(x-3y) = 4x^2 - 12xy$
5. $3ab\,(2a - 3ab - 21c) = 6a^2b - 9a^2b^2 - 63abc$
6. $-\frac{1}{2}x\,(xy - x^2 + 3y^2 - 6xy^2) = -\frac{1}{2}x^2y + \frac{1}{2}x^3 - 1\frac{1}{2}xy^2 + 3x^2y^2$

If both factors consist of a number of terms, *multiply each of the terms of one factor by each term of the other factor.* Simplify the

expression thus obtained by adding the partial products algebraically, if possible.

Examples:

1. $(a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$
2. $(a + b)(a - b) = a^2 + ab - ab - b^2 = a^2 - b^2$
3. $(a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$
4. $(3x - 2y)(5x + 4y) = 15x^2 - 10xy + 12xy - 8y^2 = 15x^2 + 2xy - 8y^2$
5. $(2a - 3b + 4c)(a + b + c) = 2a^2 - 3ab + 4ac + 2ab - 3b^2 + 4bc + 2ac - 3bc + 4c^2 = 2a^2 - 3b^2 + 4c^2 - ab + 6ac + bc$
6. $(2a - 2b)^2 = (2a - 2b)(2a - 2b) = 4a^2 - 4ab - 4ab + 4b^2 = 4a^2 - 8ab + 4b^2$

Rules for Division in Algebra. The following are the fundamental rules for division in algebra:

Rule 1: *The coefficient of the quotient equals the coefficient of the dividend divided by the coefficient of the divisor.*

Rule 2: *The exponent of a letter in the quotient equals the exponent of the same letter in the dividend, minus the exponent of the letter in the divisor.*

Rule 3: *Like signs in dividend and divisor produce a positive* (+) *quotient. Unlike signs produce a negative* (−) *quotient.*

Examples:

1. $6ab \div 3 = 2ab$
2. $6a^2b^2 \div 3ab = 2a^{2-1}b^{2-1} = 2ab$
3. $15a^5 \div 5a^2 = 3a^3$
4. $-a^4 \div a^2 = -a^2$
5. $a^4 \div -a = -a^3$
6. $-a^5 \div -a^4 = +a^{5-4} = a$
7. $a^3 \div a^3 = a^{3-3} = a^0 = 1$

We know from the rules of arithmetic that when dividend and divisor are alike, the quotient equals 1; this last example, therefore, indicates that any quantity the exponent of which is (^{0}) is equal to 1.

8. $\dfrac{12x^5y^4t^3}{-6x^2yt^3} = -2x^3y^3$

It will be remembered from the rules for arithmetic that a fraction line is equivalent to a division sign.

When the dividend consists of more than one term, but the divisor of one term only, divide each of the terms of the dividend by the divisor.

Examples:

1. $(6x^2y^2 - 4x^2y + 2xy^2) \div 2xy = 3xy - 2x + y$
2. $(-39a^3b^2 + 18a^2b^2 - 27ab^2) \div -3ab^2 = 13a^2 - 6a + 9$
3. $\dfrac{m^4n^3 - m^3n^2 + 4m^2n^2}{4mn^2} = \frac{1}{4}m^3n - \frac{1}{4}m^2 + m$

Dividend and Divisor Consisting of Several Terms. When both the dividend and divisor consist of several terms, the division can be worked out only if the divisor is a factor of the dividend, which in practical problems is not often the case. The method followed, while of little practical value, is indicated below. The rules of procedure are:

1. Arrange dividend and divisor according to the power of some one letter.
2. Divide the first term of the dividend by the first term of the divisor. This gives the first term of the quotient.
3. Multiply all the terms of the divisor by the first term of the quotient, just found; subtract this product from the dividend.
4. The remainder is regarded as a new dividend. Its first term is divided by the first term of the divisor, to obtain the second term in the quotient.
5. Multiply all the terms of the divisor by the second term of the quotient; subtract this product from the first remainder.
6. Continue this procedure until the remainder becomes 0. If no such remainder is obtained, but instead a remainder is found the first term of which cannot be divided by the first term of the divisor, then the division cannot be carried out, because in that case the divisor is not a factor of the dividend.

In this calculation, arrange the terms as below:

$$\text{Dividend} \;\left|\; \begin{array}{l} \text{divisor} \\ \hline \text{quotient} \end{array}\right.$$

As an example divide:

$$(x^2 - 72 + x) \div (9 + x)$$

Arrange according to the power of x, and write out as shown above:

$$\begin{array}{rrrr|l}
 & x^2 + & x & - 72 & x + 9 \\ \cline{5-5}
\text{Subtract} & x^2 + & 9x & & x - 8 \\ \cline{2-4}
\text{First remainder} & & -8x & - 72 & \\
\text{Subtract} & & -8x & - 72 & \\ \cline{3-4}
 & & 0 & 0 &
\end{array}$$

Factoring. Factoring, in algebra, is a most important operation, because a great number of the calculations with letters are

carried out merely by factoring and cancellation. The rules for cancellation are identical with those in arithmetic, like factors being cancelled. For factoring in algebra, however, a number of different rules must be applied.

Any simple algebraic quantity of more than one factor can always be resolved into its factors. For example $9a^2x^2 = 9 \times ax \times ax$, or $9 \times a^2 \times x^2$, or $9 \times a \times a \times x \times x$.

When an expression consists of several terms, all of which have a common factor, the expression may be resolved into two factors by dividing each term by the common factor. It is evident that the factoring can be proved by multiplying together the factors, which will result in the given quantity as a product.

Examples:

1. Factor $27x^2y - 18xy^2 + 12x^2y^2$
   The common factor is $3xy$. Dividing all the terms by this we have as the factors:
   $$3xy\ (9x - 6y + 4xy)$$
2. $16m^2n^2 - 12mn = 4mn\ (4mn - 3)$
3. $2abc - 4ab - 6bc = 2b\ (ac - 2a - 3c)$
4. $3a^3 + 3a^2 + 3a = 3a\ (a^2 + a + 1)$

Factors of Common Algebraic Expressions. There are a number of algebraic expressions the factors of which should be committed to memory, because of the frequency with which they occur in calculations. The most common of these are:

1. $a^2 + 2ab + b^2 = (a + b)\ (a + b)$
2. $a^2 - 2ab + b^2 = (a - b)\ (a - b)$
3. $a^2 - b^2 = (a + b)\ (a - b)$

Note that in Examples (1) and (2) the first and last terms are the *squares* of terms a and b and the second or middle term equals *twice the product* of terms a and b. The sign of the middle term depends on the sign between a and b in the factors. In Example (3), where the factors are $(a + b)$ and $(a - b)$, there is no middle term, and b^2 is negative. The following expressions can be resolved into factors in a similar manner:

$$4a^2 + 8ab + 4b^2 = (2a + 2b)\ (2a + 2b)$$
$$4a^2 - 8ab + 4b^2 = (2a - 2b)\ (2a - 2b)$$
$$4a^2 - 4b^2 = (2a + 2b)\ (2a - 2b)$$

and still further:

$$36m^2 + 60mn + 25n^2 = (6m + 5n)\ (6m + 5n)$$
$$16x^2 - 24xy + 9y^2 = (4x - 3y)\ (4x - 3y)$$
$$49t^2 - 4s^2 = (7t + 2s)\ (7t - 2s)$$

As the figure 1 is the square of 1, we have according to the same rules:

$$x^2 + 2x + 1 = (x + 1)(x + 1)$$
$$x^2 - 2x + 1 = (x - 1)(x - 1)$$
$$x^2 - 1 = (x + 1)(x - 1)$$

To Determine when Expression is not a Perfect Square. In order to ascertain whether an expression of three terms can be resolved into factors as indicated in the preceding paragraph, note first whether the coefficients of two of the terms are squares of whole numbers, and if they have like signs. Then see if the letters of these two terms have exponents divisible by 2. If these conditions are complied with, extract the square roots of each of these two terms; then multiply together the square roots thus obtained and double the coefficient of the product. The result should equal the third term of the given expression. If not, the expression cannot be resolved into factors as indicated, or in other words, it is not a perfect square.

Example: Is $4x^2 + 16y^2 - 16xy$ a perfect square, and can it thus be resolved into two like factors?

The numbers 4 and 16 in the two first terms are squares of whole numbers and have like signs (+); the exponents of the letters in these two terms are divisible by 2. The square root of $4x^2$ is $2x$, and the square root of $16y^2$ is $4y$. Multiply these two roots together. Then, $2x \times 4y = 8xy$. Double the coefficient of this product; $2 \times 8xy = 16xy$. This result equals the third term of the given expression, hence it can be resolved into two like factors, which are $(2x - 4y)(2x - 4y)$. The minus sign in the factors is used because the term $16xy$ is preceded by a minus sign.

The difference between two quantities, each of which is a perfect square, can always be resolved into factors, as indicated by the example

$$a^2 - b^2 = (a + b)(a - b)$$

It is only necessary to determine if each of the terms is a perfect square. The square roots then take the places of a and b in the sample formula. For example:

$$81x^4y^2 - 64a^2 = (9x^2y + 8a)(9x^2y - 8a)$$

An expression of the form $a^2 + b^2$ cannot be resolved into factors.

Examples Illustrating Rules for Factoring. A number of examples are given in the following which should be carefully studied. They show the application of the various rules for factoring given in the preceding paragraphs.

Examples:

1. $6ax^2y^3 - 24ay^7 =$
$6ay^3 (x^2 - 4y^4) =$
$6ay^3 (x + 2y^2) (x - 2y^2)$

2. $160x^2y^2 - 80xy^2 + 10y^2 =$
$10y^2 (16x^2 - 8x + 1) =$
$10y^2 (4x - 1) (4x - 1)$

3. $ac + ad - bc - bd =$
$a (c + d) - b (c + d) =$
$(c + d) (a - b)$

4. $a^2 - ab - bc + ac =$
$a^2 + ac - ab - bc =$
$a (a + c) - b (a + c) =$
$(a - b) (a + c)$

5. $49m^4n^4 + 14m^2n^2 + 1 =$
$(7m^2n^2 + 1) (7m^2n^2 + 1) = (7m^2n^2 + 1)^2$

Fractions in Algebra. A fraction, in algebra, is any expression in which a fraction line is used to indicate a division. In fact, the division sign is seldom used in algebra, but the dividend is generally written as the numerator and the divisor as the denominator of a fraction. For example, instead of writing $(6a + 3b) \div 5c$, we write:

$$\frac{6a + 3b}{5c}$$

Algebraic fractions can be reduced to their simplest form by cancellation of like factors, the same as in arithmetic. Note that expressions enclosed within parentheses, as $(x + y)$, are to be considered as single letters or symbols in all cases involving cancellation.

Examples:

1. $\dfrac{6a^2}{3a} = \dfrac{2 \times 3 \times a \times a}{3 \times a} = 2a$

2. $\dfrac{6a^2b}{2ac} = \dfrac{6 \times a \times a \times b}{2 \times a \times c} = \dfrac{3ab}{c}$

3. $\dfrac{x^2 - y^2}{x - y} = \dfrac{(x + y) (x - y)}{x - y} = x + y$

4. $\dfrac{3ax^2 - 3ay^2}{6a (x + y)} = \dfrac{3a (x^2 - y^2)}{6a (x + y)} =$
$\dfrac{3a (x + y) (x - y)}{6a (x + y)} = \dfrac{x - y}{2}$

Finding the Least Common Denominator. In adding or subtracting fractions, whether in arithmetic or algebra, it is necessary that all the fractions have a common denominator. The least common denominator is found in algebra the same as in arithmetic. Resolve all the denominators into their factors. The least common denominator must contain every type or kind of factor *at least once;* and if any factor occurs in any one denominator more than once, it must be used in the least common denominator *as many times as it occurs in any one of the given denominators.*

Find the least common denominator of the fractions

$$\frac{6a}{2ab^2} \quad \frac{5b}{3a^2b} \quad \frac{5ab}{6ac}$$

Denominators:	$2ab^2$	$3a^2b$	$6ac$
Factors:	$2 \times a \times b \times b$	$3 \times a \times a \times b$	$2 \times 3 \times a \times c$

The least common denominator must contain the factors 2; 3; a (two times); b (two times); and c. Hence, the least common denominator is:

$$2 \times 3 \times a \times a \times b \times b \times c = 6a^2b^2c$$

Examples:

1. Denominators: $6a(1-b)$ $(1-b)^2$
   Factors: $6 \times a \times (1-b)$ $(1-b)(1-b)$
   Least common denominator: $6a(1-b)(1-b) = 6a(1-b)^2$
2. Denominators: $2a^2b^3$ $3ab^3c$
   Factors: $2 \times a \times a \times b \times b \times b$ $3 \times a \times b \times b \times b \times c$
   Least common denominator: $2 \times 3 \times a \times a \times b \times b \times b \times c = 6a^2b^3c$

Addition and Subtraction of Fractions. When the method for obtaining the least common denominator is understood, the addition and subtraction of fractions is a simple matter. After the least common denominator has been found, multiply the numerator of each fraction by the quotient obtained by dividing the least common denominator by the original denominator of each fraction, the same as in arithmetic.

Example:

$$\frac{6a}{2ab^2} + \frac{5b}{3a^2b} + \frac{5ab}{6ac}$$

The least common denominator is $6a^2b^2c$. The numerator of the first fraction will be multiplied by:

$$\frac{6a^2b^2c}{2ab^2} = 3ac$$

Hence, $6a \times 3ac = 18a^2c$

Second numerator is multiplied by:

$$\frac{6a^2b^2c}{3a^2b} = 2bc$$

Hence, $5b \times 2bc = 10b^2c$

Third numerator is multiplied by:

$$\frac{6a^2b^2c}{6ac} = ab^2$$

Hence $5ab \times ab^2 = 5a^2b^3$

The whole expression reduced to a common denominator is then:

$$\frac{18a^2c + 10b^2c + 5a^2b^3}{6a^2b^2c}$$

In the example given, further simplification is impossible, because the terms in the numerator cannot be added to make the expression more compact. Wherever the terms in the numerator can be added, this is done after the terms are reduced to a common denominator.

Examples:

1. $\frac{a}{2} + \frac{a}{3} + \frac{1}{a}$

The least common denominator $= 2 \times 3 \times a = 6a$.

The fractions reduced to the least common denominator are:

$$\frac{3a^2}{6a} + \frac{2a^2}{6a} + \frac{6}{6a} = \frac{5a^2 + 6}{6a}$$

2. $\frac{2a-1}{5} + \frac{3a-2}{6} + \frac{4a-3}{3}$

The least common denominator $= 2 \times 3 \times 5 = 30$. Hence:

$$\frac{12a - 6}{30} + \frac{15a - 10}{30} + \frac{40a - 30}{30} = \frac{67a - 46}{30}$$

3. $\frac{3a - 4b}{6} - \frac{3a - 4b}{8} =$

$$\frac{12a - 16b}{24} - \frac{9a - 12b}{24} =$$

$$\frac{12a - 16b - (9a - 12b)}{24} =$$

$$\frac{12a - 16b - 9a + 12b}{24} = \frac{3a - 4b}{24}$$

Multiplication of Fractions. The rule for multiplication in algebra is identical with the rule used in arithmetic: *The product*

of the numerators of the factors equals the numerator, and the product of the denominators of the factors equals the denominator of the result.

Examples:

1. $\frac{a}{5} \times \frac{b}{6} = \frac{ab}{30}$

2. $\frac{a^2b}{c} \times \frac{ab^2}{d} = \frac{a^3b^3}{cd}$

3. $\frac{6a}{5} \times \frac{3b}{d} = \frac{18ab}{5d}$

Cancellation of like factors in numerator and denominator simplifies the calculation, the same as in arithmetic.

4. $\frac{6a}{b} \times \frac{3b}{c} \times \frac{2c}{9a} = \frac{2 \times 3 \times 6abc}{9abc} = 4$

5. $\frac{3xy^2t}{5x^2y} \times \frac{5t}{6xy} = \frac{3 \times 5xy^2t^2}{5 \times 6x^3y^2} = \frac{t^2}{2x^2}$

6. $\frac{2\,(x^2 - 1)}{3} \times \frac{18}{7\,(x + 1)} = \frac{2 \times 18 \times (x + 1)\,(x - 1)}{3 \times 7\,(x + 1)} = \frac{12\,(x - 1)}{7}$

The last example indicates that quantities consisting of more than one term should be factored, if possible, to permit cancellation.

Division of Fractions. The division of fractions in algebra follows the identical rule used in arithmetic: *Invert the divisor, and proceed as in multiplication.*

Examples:

1. $\frac{3a}{7} \div \frac{5a}{3} = \frac{3a}{7} \times \frac{3}{5a} = \frac{9a}{35a} = \frac{9}{35}$

2. $\frac{6x^2y}{5a^3b} \div \frac{9xy^3}{5a^4b^2} = \frac{6x^2y}{5a^3b} \times \frac{5a^4b^2}{9xy^3} = \frac{5 \times 6x^2ya^4b^2}{5 \times 9xy^3a^3b} = \frac{2xab}{3y^2}$

3. $\left(a + \frac{b}{c}\right) \div \frac{a}{b} = \left(\frac{ac}{c} + \frac{b}{c}\right) \div \frac{a}{b} = \frac{ac + b}{c} \times \frac{b}{a} = \frac{(ac + b)\,b}{ac} = \frac{abc + b^2}{ac}$

Square Roots. From the rules given for positive and negative quantities we have:

$$(+\,a) \times (+\,a) = +\,a^2, \text{ and } (-\,a) \times (-\,a) = +\,a^2$$

From this we can draw the conclusion that $\sqrt{a^2}$ may equal either $+\,a$ or $-\,a$. In fact, every quantity has algebraically two square

roots, numerically equal, but one positive and one negative. The significance of this will be shown in the study of quadratic equations. The fact that a root can be either positive or negative is indicated by using the sign $\pm$. Hence, $\sqrt{4} = \pm 2$.

The square root of an algebraic quantity is extracted as follows: Extract the square root of the numerical coefficients as in arithmetic, and divide the exponent of each letter under the square root sign by 2. Prefix the sign $\pm$ to the root thus obtained.

Examples:

1. $\sqrt{16a^2b^2} = \pm 4ab$

2. $\sqrt{25a^6b^4c^2} = \pm 5a^3b^2c$

3. $\sqrt{\dfrac{36a^4}{b^2}} = \pm \dfrac{6a^2}{b}$

It is not always possible to extract the square root of the complete quantity beneath the square root sign, and yet retain whole numbers as exponents. In such cases part of the expression may be left under the root sign, while the square root is extracted of the remainder.

Examples:

1. $\sqrt{16a^2b} = \pm 4a\sqrt{b}$

2. $\sqrt{\dfrac{16a^2}{b}} = \sqrt{\dfrac{16a^2 \times b}{b \times b}} = \sqrt{\dfrac{16a^2b}{b^2}} = \pm \dfrac{4a}{b}\sqrt{b}$

The following methods with regard to the coefficient are also used:

3. $\sqrt{125a^2b^2} = \sqrt{5 \times 25a^2b^2} = \pm 5ab\sqrt{5}$

4. $\sqrt{108a^2b} = \sqrt{3 \times 36a^2b} = \pm 6a\sqrt{3b}$

CHAPTER II

USE OF EQUATIONS IN SOLVING PROBLEMS

An *equation* is a statement of equality between two expressions. Thus, $5x = 105$, is an equation. Equations are used for the solution of mathematical problems. When a problem is presented, there is always one or more quantities to be found as an answer to the problem. These quantities are called *unknown* quantities. If there is only one unknown quantity in a problem, it is generally designated by the letter x in the equation used as an aid in the solution of the problem. If there is more than one unknown quantity, the others are designated by letters also selected at the end of the alphabet, as y, z, u, t, etc.

Different Types of Equations. An equation is said to be of the *first degree*, if it contains the unknown in the first power only; that is, if the unknown in the equation has no exponent. For example, $3x = 9$, is an equation of the first degree, because x has no exponent. Strictly speaking, the exponent is (1), but (1) is never written out as an exponent. It is well to always bear in mind, however, that x really means x^1. In the same way, note that a letter without a numerical coefficient, strictly speaking, has a coefficient equal to 1; so that, for example, x actually equals $1x$, or even $1x^1$, although *exponents and coefficients equal to 1 are never written out.*

An equation which contains the unknown in the second, or first and second, but no higher, power, is called a *quadratic* equation. Example: $x^2 + 3x = 18$.

An equation which contains the unknown in the third power is called a *cubic* equation. Example: $x^3 + 3x^2 + x = 22$.

Use of Simple Equations. The use of equations will be illustrated by examples showing their application to practical problems. Many of these problems are so simple that they can be solved readily by arithmetic without recourse to equations; but these simple examples are here used to show in a clear manner the principles involved.

Problem 1: The cost of 9 pounds of tin is \$2.61. What is the price of tin per pound?

In solving any problem by means of an equation, first determine what is the unknown quantity, and call this quantity x. In the problem given, the price per pound is x. Then insert this x into an equation, making use of the information given in the problem. In this case, we have:

$$9x = 261$$

It is evident that if the quantity to the left of the equal sign ($9x$) equals the quantity to the right of the equal sign (261), then these quantities will also be equal if both are divided by the same number. Hence,

$$\frac{9x}{9} = \frac{261}{9}$$

By cancellation, and division, this becomes:

$$x = 29$$

This is the answer to the problem. The price of one pound of tin is 29 cents.

Problem 2: It is known that $\frac{3}{7}$ of the total capacity of a water tank is 39 gallons. Find the total capacity.

Assume that the total capacity is x. Then:

$$\frac{3}{7}x = 39$$

In the same way as we can divide both sides of an equation by the same number, without disturbing the condition of equality, so we can also multiply both sides by the same number, and still have the two sides or members equal. Hence,

$$\frac{7}{3} \times \frac{3}{7}x = \frac{7}{3} \times 39$$

By carrying out the arithmetical work:

$$x = 91$$

The capacity of the water tank is 91 gallons.

From the two examples given it will be seen that the object of the division in Problem (1) and of the multiplication in Problem (2) was to obtain the unknown quantity x in a form where it would have a coefficient equal to 1. When the unknown x, with a coefficient equal to 1, is on one side of the equal sign, and only known numbers or quantities on the other, the problem is solved. The only difficulty met with in equations is to simplify the equation to this form.

Problem 3: A man working 54 hours a week pays, at the end of the week, \$7.30 out of his week's pay; he has \$8.90 left. How much does he earn per hour?

Let x cents be the earnings per hour; then $54x$ are the total earnings per week, in cents. Out of this 730 cents are paid out, and the remainder equals 890 cents. Hence,

$$54x - 730 = 890$$

To solve this equation, all the known quantities must be transposed to the right-hand side. To do this, add 730 to both members of the equation; this will not change the condition of equality.

$$54x - 730 + 730 = 890 + 730$$

By simplifying on each side of the equal sign:

$$54x = 1620$$

Now divide both sides by 54, the coefficient of x:

$$\frac{54x}{54} = \frac{1620}{54}$$

$$x = 30 \text{ cents per hour.}$$

Transposition. It will be seen that in the equation in Problem (3), the effect of adding 730 to both sides of the equation was to change its form from

$$54x - 730 = 890$$

to the form

$$54x = 890 + 730$$

This involves an important rule for the solution of equations:

Rule: Any independent term may be transposed from one side of the equal sign to the other by simply changing its sign; that is, + on one side of the equal sign becomes − on the other side, and vice versa. By *independent term* is meant one not tied to the other terms by signs or arrangements implying multiplication or division.

The following examples will show the application of the rule of transposition to equations.

Example 1:

$$\begin{aligned} 22x - 11 &= 15x + 10 \\ 22x - 15x &= 10 + 11 \\ 7x &= 21 \\ x &= 3 \end{aligned}$$

A term, as $15x$, to the right in the first line, not preceded by any sign, is always assumed to be preceded by a + sign, or to be *positive.* Hence, when this term was transposed to the left-hand side, it became a *negative* term, or preceded by a − sign.

Example 2:

$$12x - 93 - (3x + 1) = 12$$

When a $-$ sign precedes a parenthesis, the parenthesis may be removed, if the signs of all the terms within the parenthesis are changed. Hence,

$$12x - 93 - 3x - 1 = 12$$

and by transposing all known terms to the right side of the equation:

$$12x - 3x = 12 + 93 + 1$$
$$9x = 106$$
$$x = \frac{106}{9} = 11\tfrac{7}{9}$$

General Rule for Solving Equations of the First Degree. We may now formulate the following rule for the solving of equations of the first degree with one unknown quantity:

Clear the equation of all fractions and parentheses. Transpose all the terms containing the unknown x to one side of the equal sign, and all the other terms to the other side. Simplify the expressions as far as possible by addition or subtraction of like terms. Then divide both sides by the coefficient of the unknown x.

A number of simple examples will be given first to show the general method of procedure.

Example 1: If it takes 18 days to assemble 4 machines, how many days would be required to assemble 14 machines?

$x =$ the number of days required.

The problem is one of proportion, and

$$x : 14 = 18 : 4, \text{ or}$$
$$\frac{x}{14} = \frac{18}{4}$$

Multiply both sides by 14 and cancel; this will give the equation a form in which x has a coefficient equal to 1:

$$\frac{x}{14} \times 14 = \frac{18}{4} \times 14$$
$$x = 63 \text{ days.}$$

Example 2: Two trains start simultaneously from two terminals 360 miles apart, traveling toward each other. One train averages 50 miles an hour, the other 30 miles an hour. How soon do they meet?

The trains will meet after x hours. The faster train will then have traveled $50x$ miles, the slower, $30x$ miles. The total distance traveled by the two trains is equal to the distance between terminals. Hence,

$$50x + 30x = 360$$
$$80x = 360$$
$$x = \frac{360}{80} = 4\tfrac{1}{2} \text{ hours.}$$

Example 3: One thousand dollars are to be divided among four persons, A, B, C and D, so that A gets \$50 more than B, and C gets \$150 more than D. The shares of B and D are to be equal. Find the share of each.

B and D will have x dollars each.

A will receive $x + 50$ dollars.

C will receive $x + 150$ dollars.

The sum of the four shares is 1000 dollars. Hence,

$$x + x + x + 50 + x + 150 = 1000$$
$$4x + 200 = 1000$$
$$4x = 1000 - 200$$
$$4x = 800$$
$$x = 200$$

A's share is $200 + 50 = 250$; B's, 200; C's, $200 + 150 = 350$; and D's, 200 dollars.

Clearing Fractions and Simplifying. The following rules will aid in clearing the fractions and simplifying the equation:

Rule 1: *When all the terms in an equation have been reduced to the same denominator, this denominator can be canceled in all the terms.* Thus,

$$\frac{5}{x-1} + \frac{2x}{x-1} = \frac{3x}{x-1}$$

Cancel the denominator $x - 1$, because it appears in *all* the terms. Then:

$$5 + 2x = 3x;\ 5 = 3x - 2x;\ 5 = x$$

Rule 2: *A term which divides all the terms on one side of the equal sign may be transposed to the other side, if it is made to multiply all the terms on that side.*

Examples:

1. $\dfrac{5}{2x} + \dfrac{3}{2x} = 9$

$$5 + 3 = 9 \times 2x;\ 8 = 18x,\ x = \frac{8}{18} = \frac{4}{9}$$

2. $\dfrac{3x}{5} + \dfrac{9x+1}{5} = 9 + x$

$$3x + 9x + 1 = 5 \times 9 + 5x;\ 7x = 44;\ x = 6\tfrac{2}{7}$$

Rule 3: A term which multiplies all the terms on one side of the equal sign may be transposed to the other side, if it is made to divide all the terms on that side.

Examples:

1. $5(x+1)+5(x+2)=15+25x$

$$(x+1)+(x+2)=\frac{15}{5}+\frac{25x}{5}$$

$$x+1+x+2=3+5x;\ 0=3x;\ x=0$$

2. $2x(x-3)=4x-2x^2$

$$x-3=\frac{4x}{2x}-\frac{2x^2}{2x}$$

$$x-3=2-x;\ 2x=5;\ x=2\tfrac{1}{2}$$

The rule that any term preceded by a + sign on one side of the equal sign may be transposed to the other side if the sign is changed to −, and that any term preceded by a − sign may be transposed to the other side if the sign is changed to +, has already been referred to in connection with "Transposition." By means of the rules given, the following examples can be solved. Always remember that the object of all operations with an equation is to *obtain the unknown x on one side of the equal sign, with a coefficient equal to 1 (that is, apparently without a coefficient), and all the known quantities on the other side.*

Examples:

1. $3x+2x-4=21$
   $3x+2x=21+4$
   $5x=25$
   $x=5$

2. $15x=57-9x+5x$
   $15x+9x-5x=57$
   $19x=57$
   $x=3$

3. $19x-17-3x+72=12+9x-25+4x+83$
   $19x-3x-9x-4x=12-25+83+17-72$
   $3x=15$
   $x=5$

4. $36=-9x$

$$\frac{36}{-9}=x$$

$$-4=x$$

5. $6x=-4$

$$x=\frac{-4}{6}$$

$$x=-\frac{2}{3}$$

6. $\frac{x}{5}=8$

$$\frac{5x}{5}=5\times 8$$

$$x=40$$

7. $5 \div x = 8$

$$\frac{5}{x} = 8$$

$$\frac{5x}{x} = 8x$$

$$5 = 8x$$

$$5/8 = x$$

8. $7(x - 2) = 35$

$$7x - 14 = 35$$

$$7x = 35 + 14$$

$$7x = 49$$

$$x = 7$$

9. $b(x - a) = c$

$$bx - ab = c$$

$$bx = c + ab$$

$$x = \frac{c + ab}{b}$$

10. $9(x + 2) - 3(x - 2) = 12(x - 4)$

$$9x + 18 - 3x + 6 = 12x - 48$$

$$18 + 6 + 48 = 12x - 9x + 3x$$

$$72 = 6x;\ 12 = x$$

11. $\dfrac{51}{5 - x} - 3 = 14$

$$\frac{51}{5 - x} - \frac{3(5 - x)}{5 - x} = \frac{14(5 - x)}{5 - x}$$

$$51 - 3(5 - x) = 14(5 - x)$$

$$51 - 15 + 3x = 70 - 14x$$

$$3x + 14x = 70 - 51 + 15$$

$$17x = 34;\ x = 2$$

12. $\dfrac{5}{x - 3} = \dfrac{3}{5 - x}$

$$\frac{5(5 - x)}{(x - 3)(5 - x)} = \frac{3(x - 3)}{(5 - x)(x - 3)}$$

$$5(5 - x) = 3(x - 3)$$

$$25 - 5x = 3x - 9$$

$$25 + 9 = 3x + 5x$$

$$34 = 8x;\ 4\tfrac{1}{4} = x$$

Note: When letters are used to represent *known* quantities, they are usually selected at the beginning of the alphabet, as a, b, c, etc.

Examples for Practice. Examples will now be given without the method of solution. Opposite each example the required value of x is given.

1. $3 = x - 7$ Answer: $x = 10$
2. $7x = -4.2$ Answer: $x = -0.6$
3. $9x + 4 = 31$ Answer: $x = 3$
4. $8x = 11 - 5x + 2x$ Answer: $x = 1$
5. $10x + 29 = 13 + 17x + 18 - 2x - 97$ Answer: $x = 19$

6. $7 = \frac{x}{3}$ Answer: $x = 21$

7. $16 = x \div \frac{5}{8}$ Answer: $x = 10$

In Example 7, remember that $x \div \frac{5}{8} = x \times \frac{8}{5} = \frac{8}{5}x$

Hence $\frac{8}{5}$ is the coefficient of x

8. $5 - \frac{x}{7} = 4$ Answer: $x = 7$

9. $6(x - 9) = 27$ Answer: $x = 13.5$

10. $a(a + x) = b$ Answer: $x = \frac{b - a^2}{a}$

11. $5(x + 1) - 2(3x - 1) = x$ Answer: $x = 3\frac{1}{2}$

12. $\frac{x}{3} + \frac{x}{4} = 14$ Answer: $x = 24$

13. $\frac{5x - 4}{4x - 5} = \frac{4}{5}$ Answer: $x = 0$

14. $\frac{87}{3x - 1} + 17 = 20$ Answer: $x = 10$

Problems Leading to Equations with One Unknown Quantity. *Problem* 1: A certain work is to be done by three men, A, B and C. Alone, A could do the work in 10 hours, B in 12 hours, and C in 15 hours. How long a time will be required when they work together?

Total number of hours required $= x$.

A completes in one hour $\frac{1}{10}$ of the work; thus, in x hours, $\frac{1}{10}x$. Similarly, B completes in x hours $\frac{1}{12}x$, and C, $\frac{1}{15}x$ of the work; but their combined work completes the job. Hence,

$$\frac{1}{10}x + \frac{1}{12}x + \frac{1}{15}x = 1$$

$$\frac{6}{60}x + \frac{5}{60}x + \frac{4}{60}x = 1$$

$$6x + 5x + 4x = 60$$

$$15x = 60$$

$$x = 4 \text{ hours}$$

Problem 2: In a triangle, the sum of the lengths of the three sides is 17 feet. The first side is 2 feet shorter than the second and $\frac{3}{4}$ of the length of the third. Find the length of the sides.

Length of third side $= x$
Length of first side $= \frac{3}{4}x$
Length of second side $= \frac{3}{4}x + 2$
Then,

$$x + \frac{3}{4}x + \frac{3}{4}x + 2 = 17$$
$$2\tfrac{1}{2}x = 17 - 2 = 15$$
$$x = 6$$

The third side thus equals 6 feet.
The first side equals $\frac{3}{4}x = \frac{3}{4} \times 6 = 4\frac{1}{2}$ feet.
The second side equals $\frac{3}{4}x + 2 = 4\frac{1}{2} + 2 = 6\frac{1}{2}$ feet.

Equations with Two Unknown Quantities. When an equation contains two unknown quantities, the values of these quantities can be determined only if *two* independent equations are given, each containing the unknown quantities. It is common practice to call the unknown quantities x and y. Equations with two unknown quantities are solved by so combining them that an equation with one unknown is found, which can be solved by the methods already explained.

Solve x and y in the equations:

$$5x + 3y = 22$$
$$3x - y = 2$$

In one of the equations, solve the value of one of the unknown, in terms of the other. Insert this value in the other equation, thus obtaining an equation with one unknown.

Following this rule, solve for y in the second equation, which is evidently the simplest method, as y in this equation has no coefficient (or, rather, a coefficient equal to 1).

$$3x - y = 2$$
$$3x - 2 = y$$

Insert this value of y in place of y in the first equation:

$$5x + 3(3x - 2) = 22$$
$$5x + 9x - 6 = 22$$
$$14x = 28$$
$$x = 2$$

Now insert this value of x in the expression for y previously given:

$$3x - 2 = y$$
$$3 \times 2 - 2 = y$$
$$4 = y$$

Hence, $x = 2$, and $y = 4$

To prove the calculation, insert the values found in one of the original equations:

$$5x + 3y = 22$$
$$5 \times 2 + 3 \times 4 = 22$$
$$10 + 12 = 22$$

This proves that the values found are correct, and that no error has been made in the calculations.

Another method is as follows: *Solve for one of the unknown in both equations, and place the quantities thus found in a new equation, from which the other unknown can then be solved.*

Apply this rule to the same equations as before.

$$5x + 3y = 22$$
$$3x - y = 2$$

Solve for y in the first equation:

$$5x + 3y = 22$$
$$3y = 22 - 5x$$
$$y = \frac{22 - 5x}{3}$$

Now solve for y in the second equation:

$$3x - y = 2$$
$$3x - 2 = y$$

Here we have now two expressions both of which are equal to y. Hence, these expressions must themselves be equal:

$$3x - 2 = \frac{22 - 5x}{3}$$

In this equation solve for x.

$$3(3x - 2) = 22 - 5x$$
$$9x - 6 = 22 - 5x$$
$$9x + 5x = 22 + 6$$
$$14x = 28$$
$$x = 2$$

Then, $y = 4$, is found as in the preceding case

There is still a third method that may often be used to advantage. *Select the unknown quantity to be eliminated from both equations. Multiply one equation by such a factor that the coefficients of the unknown to be eliminated will be numerically equal. If the unknown terms to be eliminated have like signs in the two equations, subtract one equation from the other; if they have unlike signs, add one to the other.*

$$5x + 3y = 22$$
$$3x - y = 2$$

Select y as the unknown to be eliminated, as it will be seen that by multiplying the second equation by 3, the y-terms will then have numerically equal coefficients. Multiply the second equation by 3:

$$3 \times 3x - 3 \times y = 2 \times 3$$
$$9x - 3y = 6$$

Now place this equation beneath the first, and add the two together.

$$\begin{aligned} 5x + 3y &= 22 \\ 9x - 3y &= 6 \\ \hline 14x \qquad &= 28 \\ x &= 2 \end{aligned}$$

The unknown y is now found in the same manner as before, by inserting the value of x just found into any of the given equations and solving for y.

$$\begin{aligned} 5 \times 2 + 3y &= 22 \\ 3y &= 22 - 10 \\ 3y &= 12 \\ y &= 4 \end{aligned}$$

Examples for Practice. Some of the examples in the following are partly worked out to indicate the method used.

1. $$\begin{aligned} x + y &= 15 \\ x - y &= 6 \\ \hline 2x \qquad &= 21 \\ x &= 10.5 \\ 10.5 + y &= 15 \\ y &= 15 - 10.5 \\ y &= 4.5 \end{aligned}$$

2. $$\begin{aligned} x + 3y &= 20 \\ 2x - 5y &= 29 \end{aligned}$$
   Then:
   $$\begin{aligned} 2x + 6y &= 40 \\ 2x - 5y &= 29 \\ \hline 11y &= 11 \\ y &= 1; \; x = 17 \end{aligned}$$

3. $$\begin{aligned} 5(x - 2) - 3(y - 1) &= 0 \\ 2(x - 2) + 7(y - 1) &= 0 \end{aligned}$$
   Multiply, to eliminate parentheses:
   $$\begin{aligned} 5x - 10 - 3y + 3 &= 0 \\ 2x - 4 + 7y - 7 &= 0 \end{aligned}$$
   Transpose unknown quantities to one side:
   $$\begin{aligned} 5x - 3y &= 10 - 3 = 7 \\ 2x + 7y &= 4 + 7 = 11 \end{aligned}$$
   Answer: $x = 2$; $y = 1$

4. $$\begin{aligned} x + 36y &= 450 \\ 36x + y &= 660 \end{aligned}$$
   Answer: $x = 18$; $y = 12$

5. $$\begin{aligned} 1.5x + 3.5y &= 33 \\ x + 2y &= 20 \end{aligned}$$
   Answer: $x = 8$; $y = 6$

6. $\dfrac{6}{x} - \dfrac{3}{y} = 4$

$\dfrac{8}{x} + \dfrac{15}{y} = -1$ Answer: $x = 2$; $y = -3$

7. $\dfrac{3}{x} + \dfrac{5}{y} = 2$

$\dfrac{9}{x} - \dfrac{10}{y} = 1$

In this case apply the method of adding the two equations, it being easily seen by inspection that the y-terms can be made alike by multiplying the first equation by 2. Hence, multiply the first equation by 2, and add the equations:

$$\frac{6}{x} + \frac{10}{y} = 4$$
$$\frac{9}{x} - \frac{10}{y} = 1$$
$$\frac{6}{x} + \frac{9}{x} = 5$$

Clearing of fractions:

$$6 + 9 = 5x;\ 15 = 5x;\ x = 3, \text{ and } y = 5$$

Problems Leading to Equations with Two Unknown Quantities. *Problem* 1: The sum of two numbers is 1000. The difference between the numbers is 222. Find the numbers.

The numbers to be found are x and y.

$$x + y = 1000$$
$$x - y = 222$$

Answer: $x = 611$; $y = 389$

Problem 2: A is 27 years older than B. Ten years ago he was 10 times as old as B. How old are A and B?

A is x years old; B, y years. Hence:

$$x - 27 = y$$

But 10 years ago, A's age was $(x - 10)$, and B's, $(y - 10)$. Hence:

$$x - 10 = 10 \times (y - 10)$$

From these two equations we find $x = 40$; $y = 13$.

Problem 3: A tank of 120 gallons capacity can be filled from two pipes; if one pipe is open for 6 minutes, and the other for 3 minutes, 100 gallons will enter the tank. If the first pipe is open 3 minutes and the second 6 minutes, 110 gallons will enter the tank. How long a time would be required for each of the pipes to fill the tank?

One pipe requires x minutes, the other y minutes.

In one minute the first pipe delivers $\frac{120}{x}$ gallons.

In one minute the second pipe delivers $\frac{120}{y}$ gallons.

Hence:

$$\frac{120}{x} \times 6 + \frac{120}{y} \times 3 = 100$$
$$\frac{120}{x} \times 3 + \frac{120}{y} \times 6 = 110$$

Multiply the first equation by 2, and subtract the second equation from the first:

$$\frac{120 \times 12}{x} + \frac{120 \times 6}{y} = 200$$
$$\frac{120 \times 3}{x} + \frac{120 \times 6}{y} = 110$$
$$\frac{120 \times 12}{x} - \frac{120 \times 3}{x} = 90$$
$$1440 - 360 = 90x; \; x = 12; \; y = 9$$

Solving Quadratic Equations. A *quadratic* equation is one in which the unknown quantity is contained in the second, or first and second, power. The following equations are, therefore, examples of quadratic equations:

$$x^2 - 27 = 39$$
$$3x^2 - 5x = 12$$

A quadratic equation is frequently called an equation of the *second degree.*

A *pure quadratic* equation is one which contains the unknown in the second power only, as $x^2 - 12 = 4$.

An *affected quadratic* equation is one which contains the unknown in both the first and second power, as $x^2 - 3x = 4$.

Solving Pure Quadratic Equations. A pure quadratic equation can always be simplified so that it takes the form $x^2 = a$; that is, the unknown quantity in the second power, without a coefficient, will be on one side of the equal sign, and the known quantities, reduced to their simplest form, will be on the other side.

To solve the equation when in this form, it is only necessary to extract the square root on each side of the equal sign. Remember that roots have both positive and negative signs, or that the square root of $a^2 = \pm a$.

Examples:

1. $x^2 = 16$
$\sqrt{x^2} = \sqrt{16}$
$x = \pm 4$

2. $3x^2 = 75$
$x^2 = 25$
$x = \pm 5$

3. $7x^2 - 8 = 9x^2 - 10$
$10 - 8 = 9x^2 - 7x^2$
$2 = 2x^2$
$1 = x^2$
$\pm 1 = x$

4. $(3x - 4)(3x + 4) = 65$
Multiply and reduce:
$9x^2 - 16 = 65$
$9x^2 = 81$
$x^2 = 9$
$x = \pm 3$

5. $(7 + x)^2 + (7 - x)^2 = 130$
$(7 + x)(7 + x) + (7 - x)(7 - x) = 130$
Multiplying, we have:
$49 + 14x + x^2 + 49 - 14x + x^2 = 130$
$2x^2 = 130 - 49 - 49 = 32$
$x^2 = 16$
$x = \pm 4$

Solving Affected Quadratic Equations. An affected quadratic equation can be reduced to the form $x^2 + px = q$; that is, the unknown quantity in the second power has a coefficient equal to 1; the unknown in the first power has a known coefficient p; and q represents the known quantities reduced to their simplest form. The equation $x^2 - 3x = 4$ is an example of an equation reduced to the form $x^2 + px = q$. The coefficient of x in this case is (-3); hence the minus sign.

To reduce an equation to its simplest form, add all the like terms together; arrange them in the form given above, and then divide by the coefficient of x^2.

Examples:

1. Reduce $6x^2 + 7 = 3x - x^2 + 12$ to its simplest form.

$$7x^2 - 3x = 5$$
$$x^2 - \tfrac{3}{7}x = \tfrac{5}{7}$$

2. Reduce $\tfrac{1}{2}x^2 - 2x + 13 = 0$ to its simplest form.

$$\tfrac{1}{2}x^2 - 2x = -13$$
$$x^2 - \frac{2}{\tfrac{1}{2}}x = -\frac{13}{\tfrac{1}{2}}$$
$$x^2 - 4x = -26$$

When the equation has been reduced to the form $x^2 + px = q$, the value of x may be found by the formula:

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$$

It is not necessary, in order to make use of this formula, to know how it is obtained. Equations can be solved by its use quickly and with the minimum risk of error. For the benefit of those students who wish to know how the formula is obtained, this will be explained later. This formula should be committed to memory, as it is applied to all quadratic equations. Expressed as a rule the formula says: *x equals half the coefficient for the second term (p) with changed sign, plus or minus the square root of the square of half the coefficient of the second term, plus the known quantity (q).*

Remember that if the known quantity is negative, $+(-q) = -q$.

Examples:

1. $x^2 + 3x = 28$

$$x = -\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 + 28} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + 28}$$

$$x = -\frac{3}{2} \pm \sqrt{\frac{9 + 112}{4}} = -\frac{3}{2} \pm \frac{11}{2}$$

$$x = -\frac{3}{2} + \frac{11}{2} = \frac{8}{2} = 4$$

or $$x = -\frac{3}{2} - \frac{11}{2} = -\frac{14}{2} = -7$$

The two values of x found are obtained by using either the plus or minus sign of the root. Either value, if inserted in the original equation, will *satisfy* it.

$$x^2 + 3x = 28$$

Insert $x = 4$

$$16 + 12 = 28$$

Insert $x = -7$,

$$49 + 3 \times (-7) = 28$$
$$49 - 21 = 28$$

2. $x^2 - 4x = 21$

$$x = +2 \pm \sqrt{(2)^2 + 21} = 2 \pm \sqrt{25}$$
$$x = 2 \pm 5$$
$$x = 7, \text{ or } x = -3$$

It is common practice to indicate the two roots of the equation x_1 and x_2, respectively. The small numbers ($_1$) and ($_2$) are not exponents, when placed at the lower corner of the letter. Hence, we would write,

$$x_1 = 7, \text{ and } x_2 = -3$$

3. $3x^2 - 24x + 45 = 0$

$3x^2 - 24x = -45$

Divide the terms by 3, the coefficient of x^2:

$x^2 - 8x = -15$

$x = +4 \pm \sqrt{(4)^2 - 15} = +4 \pm \sqrt{16 - 15}$

$x = +4 \pm 1$

$x_1 = 5;\ x_2 = 3$

4. $(x - 3)(x - 5) = 0$

Multiply:

$x^2 - 3x - 5x + 15 = 0$

Reduce and arrange for solving:

$x^2 - 8x = -15$

$x = +4 \pm \sqrt{16 - 15}$

$x = 4 \pm 1$

$x_1 = 5;\ x_2 = 3$

Examples for Practice. The following examples in quadratic equations are intended for the use of students desiring to practice the methods of solution. The values of x are given opposite the different problems.

1. $x^2 = 81$ — Answer: $x = \pm 9$
2. $5x^2 = 125$ — Answer: $x = \pm 5$
3. $9x^2 - 36 = 5x^2$ — Answer: $x = \pm 3$
4. $7x^2 - 8 = 24 - x^2$ — Answer: $x = \pm 2$
5. $x^2 - 8x = 20$ — Answer: $x_1 = 10;\ x_2 = -2$
6. $x^2 - 8x = -12$ — Answer: $x_1 = 6;\ x_2 = 2$
7. $x^2 - x = 12$ — Answer: $x_1 = 4;\ x_2 = -3$
8. $6x^2 + 48x = 54$ — Answer: $x_1 = 1;\ x_2 = -9$

Problems Leading to Quadratic Equations. *Problem* 1: The sides enclosing the right angle in a right-angled triangle are in the proportion 12:5. The side opposite the right angle is 299 feet long. Find the length of the other sides.

In a right-angled triangle, the square of the side opposite the right angle equals the sum of the squares of the sides enclosing this angle.

One of the sides to be found $= x$; the other side to be found $= \frac{12}{5}x$.

Then:

$$x^2 + \left(\frac{12}{5}x\right)^2 = 299^2$$

Use tables in an engineering handbook to obtain squares and square roots of large numbers.

$$x^2 + \frac{144}{25}x^2 = 89{,}401$$
$$25x^2 + 144x^2 = 89{,}401 \times 25$$
$$169x^2 = 2{,}235{,}025$$
$$x^2 = \frac{2{,}235{,}025}{169} = 13{,}225$$
$$x = 115 \text{ feet}$$

In this case we do not use the negative value of the root, because, while it satisfies the *equation*, it does not apply to the *problem* here presented. We could not conceive of a side $-$ 115 feet long.

The other side then equals $\frac{12}{5} \times 115 = 276$ feet.

Problem 2: Find two factors of 96 the sum of the squares of which is 208.

The factors are x and $\frac{96}{x}\cdot$ Note that $x \times \frac{96}{x} = 96$.

$$x^2 + \left(\frac{96}{x}\right)^2 = 208$$

Note that $\left(\frac{96}{x}\right)^2 = \frac{96^2}{x^2}\cdot$

$$\frac{x^4}{x^2} + \frac{9216}{x^2} = \frac{208x^2}{x^2}$$
$$x^4 - 208x^2 = -9216$$

An equation having the unknown in the fourth and second power only can be solved just as one having the unknown in the second and first powers; only, in this case, x^2 takes the place of x in the formula for solving the unknown.

$$x^2 = +104 \pm \sqrt{104^2 - 9216}$$
$$x^2 = +104 \pm \sqrt{10{,}816 - 9216} = 104 \pm 40$$
$$x_1^2 = 144, \text{ and } x_2^2 = 64$$

Now extract the square roots of $x^2 = 144$ and $x^2 = 64$.

Then, $x_1 = \pm 12$, and $x_2 = \pm 8$.

Here both roots can be used because $12 \times 8 = 96$, and $-12 \times -8 = 96$.

Problem 3: A certain number of men agreed to pay an equal share for the purpose of buying a machine that was to cost \$6300. Two of the men later changed their mind, and the remainder each put up \$200 more than originally agreed in order to make up for the shares of these two. How many men made the first agreement?

The original number of men $= x$.

The share of each, originally $= \dfrac{6300}{x}$.

The number of men actually buying the machine $= x - 2$.

The share of each of these $= \dfrac{6300}{x-2}$.

This last share was \$200 greater than the original amounts assessed; hence:

$$\frac{6300}{x-2} - \frac{6300}{x} = 200$$
$$6300x - 6300\,(x-2) = 200x(x-2)$$
$$6300x - 6300x + 12{,}600 = 200x^2 - 400x$$
$$12{,}600 = 200x^2 - 400x$$

Divide by 200, and arrange equation like sample form:

$$x^2 - 2x = 63$$
$$x = +1 \pm \sqrt{1+63} = +1 \pm 8$$
$$x_1 = 9;\ x_2 = -7$$

The number of men was 9. The root -7 is impossible as a solution of the problem, although it mathematically satisfies the equation.

Deducing the Formula for x in a Quadratic Equation. The method by means of which the formula

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$$

is obtained from the equation $x^2 + px = q$ is as follows:

Add to the left-hand side of the equation a quantity such that this side will be a perfect square of the form $a^2 + 2ab + b^2$. The added quantity must, of course, also be added to the right side of the equation in order to retain the relation of equality.

The quantity to be added is $\dfrac{p^2}{4}$.

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} + q$$

Now, as $\left(x + \dfrac{p}{2}\right)\left(x + \dfrac{p}{2}\right) = x^2 + px + \dfrac{p^2}{4}$, we can extract the square root on both sides of the equal sign:

$$\sqrt{x^2 + px + \frac{p^2}{4}} = \sqrt{\frac{p^2}{4} + q}$$

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} + q}$$

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$$

If the given equation is of the form

$$ax^2 + bx = c,$$

then the unknown quantity x may be found directly by the formula:

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

Note that the sign of the coefficient for x in the first power (b) changes its sign, the same as in the formula already given.

Example:

$$3x^2 - 15x = 0$$

$$x = \frac{+15 \pm \sqrt{225 + 0}}{2 \times 3} = \frac{15 \pm 15}{6}$$

$$x_1 = \frac{30}{6} = 5; \; x_2 = \frac{0}{6} = 0$$

CHAPTER III

PROBLEMS INVOLVING UNKNOWN ANGLES

THE fundamental principles of geometry, trigonometry, and algebra are utilized extensively by the designers of machinery and other mechanical devices, as indicated by the number of solutions in this book based upon these branches of mathematics, since the problems were selected from actual designing practice. As wide a variety of problems as possible has been presented in this and following chapters to illustrate numerous applications of important mathematical principles and various methods of analyzing and solving different problems. All of the examples in this chapter involve determining unknown angles when certain known factors are given. By segregating the problems in this manner and by placing similar types of problems together in the same chapter, it will be comparatively easy to find whatever type of problem is wanted. This same method of classification has been applied to the other chapters as far as practicable.

Having Dimensions E, D, and Radius R, to Find Angle x, Fig. 1. It is desired to find angle x (see diagram, Fig. 1) when the lengths of lines E and D and the radius R are known.

Solution: Erect a perpendicular at F passing through point C, from which the arc is struck, and construct AB parallel with E, extending it to intersect the perpendicular. In the triangle ABC, side $AB = E$; and side $BC = D - R$.

Therefore

$$AC = \sqrt{E^2 + (D - R)^2}$$

Then

$$\text{Tan } a = \frac{BC}{AB} = \frac{D - R}{E}$$

$$\text{Sin } b = \frac{R}{AC} = \frac{R}{\sqrt{E^2 + (D - R)^2}}$$

$$c = 90 \text{ degrees} - (a + b)$$

Hence,

$$x = 180 \text{ degrees} - (90 \text{ degrees} + c)$$

Example: If $D = 3.75$ inches, $R = 2.50$ inches, and $E = 5$ inches, then

$$\text{Tan } a = \frac{3.75 - 2.5}{5} = 0.25$$

$$a = 14 \text{ degrees } 2 \text{ minutes}$$

$$\text{Sin } b = \frac{2.5}{\sqrt{5^2 + (3.75 - 2.5)^2}} = 0.48507$$

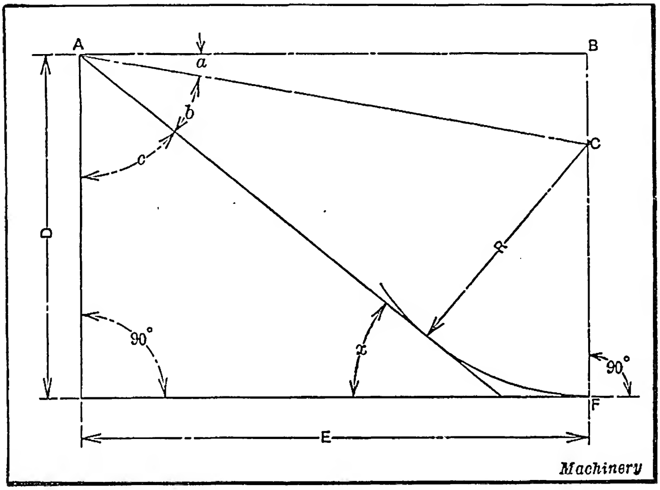

Fig. 1. Having Dimensions E, D and Radius R, to find Angle x

$$b = 29 \text{ degrees } 1 \text{ minute}$$

$$c = 90 \text{ deg.} - (14 \text{ deg. } 2 \text{ min.} + 29 \text{ deg. } 1 \text{ min.})$$
$$= 46 \text{ degrees } 57 \text{ minutes}$$

Therefore,

$$x = 180 \text{ deg.} - (90 \text{ deg.} + 46 \text{ deg. } 57 \text{ min.})$$
$$= 43 \text{ degrees } 3 \text{ minutes}$$

How to Find Angle ϕ, Fig. 2. The problem is to find the angle ϕ between line GD and a line GB which is tangent to an arc of given radius drawn from center O.

Solution: As OC is perpendicular to CD and OA is perpendicular to GB, angle ϕ = angle ω and angle ω = angle θ − angle δ. $OF = OA + BD = 0.205 + 0.040 = 0.245$ inch and $OD =$

$\sqrt{OC^2 + CD^2} = \sqrt{0.217^2 + 0.138^2} = 0.2572$ inch. Cos $\theta = \dfrac{OC}{OD} = \dfrac{0.217}{0.2572} = 0.8437$, which is the cosine of an angle of 32 degrees 28 minutes. Cos $\delta = \dfrac{OF}{OD} = \dfrac{0.245}{0.2572} = 0.95257$, which is the cosine of an angle of 17 degrees 43 minutes. Angle ω = 32 degrees 28

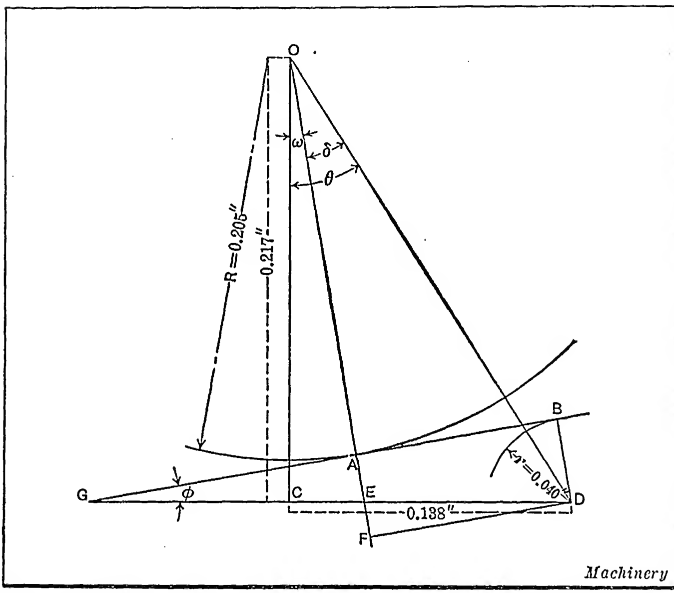

Fig. 2. To find Angle ϕ between Line GD and a Line GB which is Tangent to an Arc of Given Radius R

minutes − 17 degrees 43 minutes = 14 degrees 45 minutes, which is, therefore, the size of angle ϕ.

Another Solution: In the right triangle OCO_1 (Fig. 3) the hypotenuse $OO_1 = \sqrt{OC^2 + O_1C^2} = \sqrt{0.217^2 + 0.138^2} = 0.25716$. By construction, it is evident that triangles OAZ and O_1ZF are similar. The proportion $\dfrac{O_1Z}{OZ} = \dfrac{O_1F}{OA}$ is therefore obtained. Adding 1 to both sides,

$$\frac{O_1Z}{OZ} + 1 = \frac{O_1F}{OA} + 1$$

Changing to fractions,

$$\frac{O_1Z + OZ}{OZ} = \frac{O_1F + OA}{OA}$$

Substituting the values given,

$$\frac{0.25716}{OZ} = \frac{0.040 + 0.205}{0.205}$$

Solving, $OZ = 0.21518$.

Cos angle $AOZ = \frac{AO}{OZ}$ $= \frac{0.205}{0.21518} = 0.95270$

Angle AOZ = 17 degrees 42 minutes. The tangent of angle O_1OC $= \frac{O_1C}{OC} = \frac{0.138}{0.217} = 0.63594$. Angle O_1OC = 32 degrees 27 minutes.

Angle AOC = angle O_1OC − angle AOZ = 32 degrees 27 minutes − 17 degrees 42 minutes = 14 degrees 45 minutes. Now the angles AOC and FIO_1, or angle ϕ, are equal, since their corresponding sides are perpendicular. Therefore, angle ϕ = 14 degrees 45 minutes.

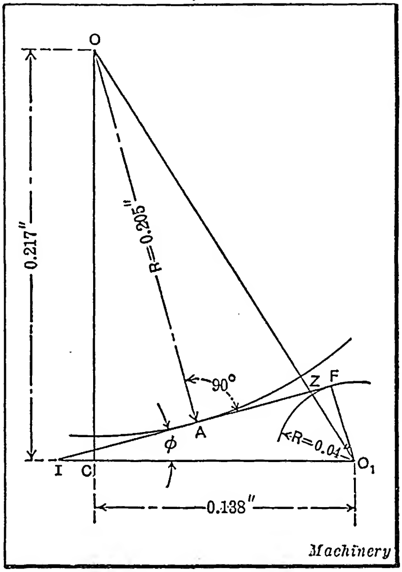

Fig. 3. Diagram illustrating Another Solution of the Problem represented by Fig. 2

To Find Angle x, Fig. 4. The problem is to find angle x in Fig. 4. This angle is required in working out the design of steel wheels.

Solution: To find angle x, first draw lines AB, AD, and EB. We now have two right-angle triangles, ACD and ECB.

Now $$AB = \sqrt{AF^2 + FB^2}$$

$$\text{Sin } a = \frac{AD}{AC} = \frac{EB}{CB}$$

$$AC + CB = AB$$

Therefore, $$\text{Sin } a = \frac{AD + EB}{AB}$$

$$\text{Sin } b = \frac{AF}{AB}$$

$$\text{Angle } e = 180 \text{ degrees} - (a + b)$$

$$x = 90 \text{ degrees} - e$$

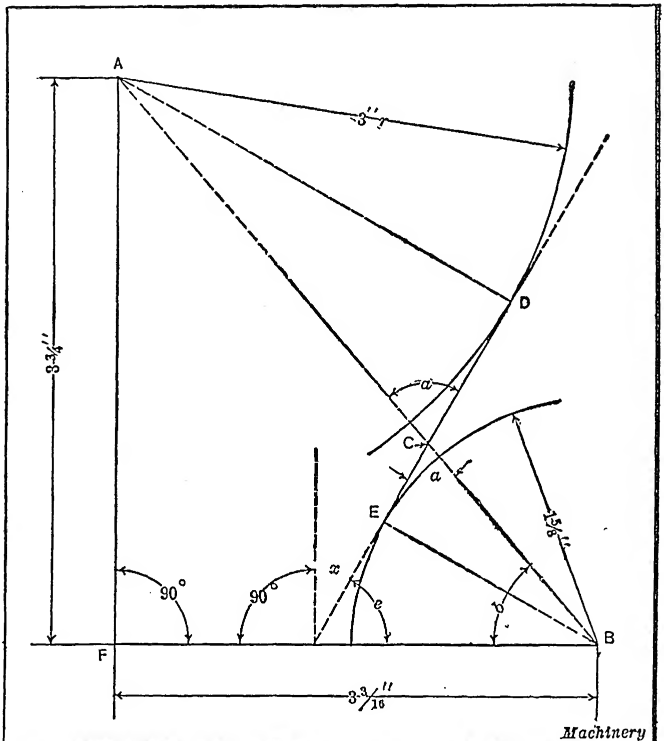

Fig. 4. To find Angle x having the Numerical Values given on the Diagram

Substituting the numerical values given in the diagram, we have

$$AB = \sqrt{3.75^2 + 3.1875^2} = 4.92165$$

$$\text{Sin } a = \frac{3 + 1.625}{4.92165} = 0.93972$$

$$a = 70 \text{ degrees } 18 \text{ seconds}$$

$$\text{Sin } b = \frac{3.75}{4.92165} = 0.76194$$

$$b = 49 \text{ degrees } 38 \text{ minutes } 9 \text{ seconds}$$

Therefore

$$e = 180 \text{ degrees} - (70 \text{ deg. } 18 \text{ sec.} + 49 \text{ deg. } 38 \text{ min. } 9 \text{ sec.})$$
$$= 60 \text{ degrees } 21 \text{ minutes } 33 \text{ seconds}$$

and

$$x = 90 \text{ degrees} - (60 \text{ deg. } 21 \text{ min. } 33 \text{ seconds})$$
$$= 29 \text{ degrees } 38 \text{ minutes } 27 \text{ seconds}$$

Tangency Problem. In Fig. 5, the problem is to find angle y, assuming that a equals 5.5 inches; b, 3 inches; and R, 1.625 inches.

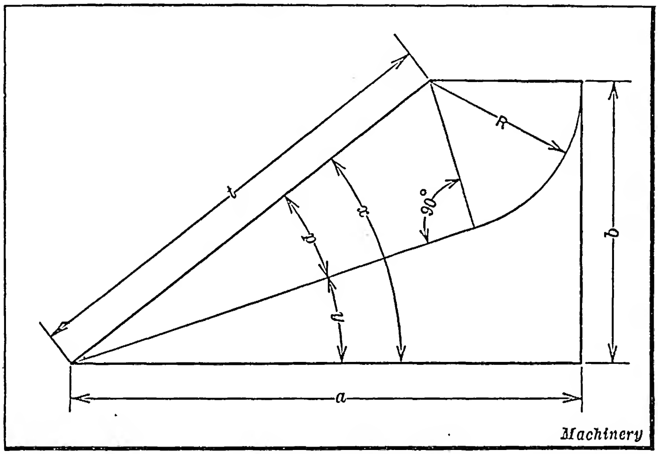

Fig. 5. To find Angle y, having Dimensions a, b, and Radius R

Solution: The following method of solving the problem stated requires only an elementary knowledge of trigonometry and the use of a table of trigonometrical functions. According to trigonometry,

$$\operatorname{Tan} x = \frac{b}{(a - R)}$$

and

$$t = \sec x \times (a - R)$$

also

$$\operatorname{Sin} p = \sin (x - y) = \frac{R}{t}$$

and

$$y = x - p$$

Substituting the known values and solving,

$$\text{Tan } x = \frac{3}{3.875} = 0.77419 = \tan 37 \text{ degrees } 44 \text{ minutes}$$

and

$$t = 1.2644 \times 3.875 = 4.89955$$

Also

$$\text{Sin } (x - y) = \frac{1.625}{4.89955} = 0.33166 = \sin 19 \text{ deg. } 22 \text{ min.}$$

Therefore,

$$y = 37 \text{ deg. } 44 \text{ min.} - 19 \text{ deg. } 22 \text{ min.} = 18 \text{ deg. } 22 \text{ min.}$$

To Calculate Angle X, Fig. 6, from the Dimensions Given. Forming tools are to be made for different sizes of poppet valve

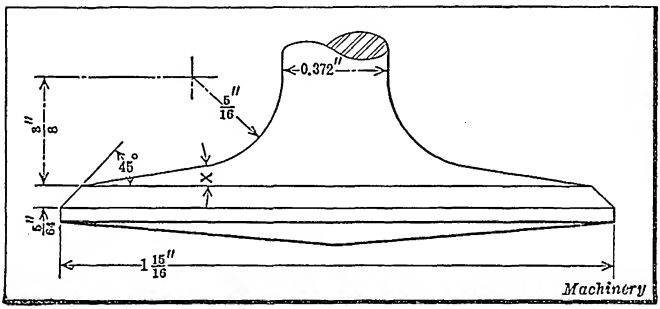

Fig. 6. To find Angle X, having the Dimensions given on the Diagram

heads and a general formula is required for finding angle X from dimensions such as given in Fig. 6.

Solution: By comparison with Fig. 6, the values for b, h, and r, Fig. 7, can be determined easily. Angle X can then be found in the following manner: In Fig. 7,

$$\text{Tan } A = \frac{h}{b} \quad (1) \qquad\qquad c = \frac{h}{\sin A} \quad (2)$$

Also,

$$c = \frac{r}{\sin B} = \frac{r}{\sin (A - X)} \tag{3}$$

From Equations (2) and (3) by comparison,

$$\frac{r}{\sin (A - X)} = \frac{h}{\sin A}$$

$$\text{Sin } (A - X) = \frac{r \sin A}{h} \tag{4}$$

From the dimensions given in Fig. 6, it is obvious that $b = 0.392125$ inch, $h = 0.375$ inch, and $r = 0.3125$ inch. Substituting these values in Equations (1) and (4) and solving, angle A will be found to be 43 degrees 43 minutes and angle $(A-X)$, to be 35 degrees, 10 minutes. By subtracting these two values, angle X will be found to equal 8 degrees 33 minutes.

To Find Angles of Special Screw Threads. The illustration Fig. 8 shows a section of a screw thread; find the angles m and n.

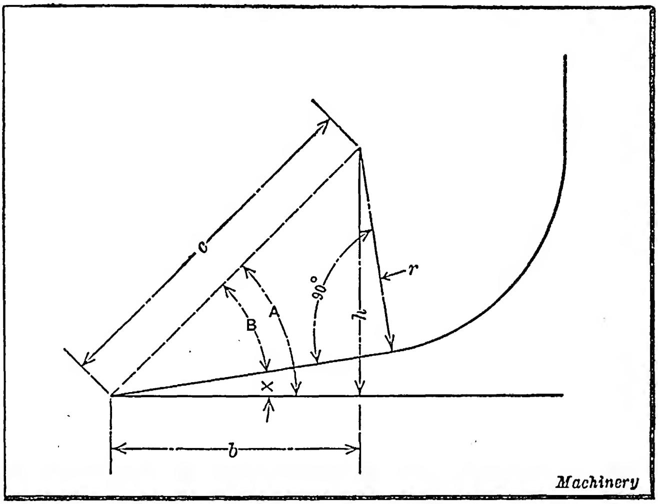

Fig. 7. Values used in determining Angle X, Fig. 6

Solution: The following method of finding the angles of a special screw thread can be applied even though the top and bottom radii of the screw thread are unequal. Draw OA and OB parallel to the sides of the thread. Then $AP = R + r$; also $BQ = R + r$.

Further, $\frac{a}{h} = \tan x$. $\frac{AP}{OP} = (R + r) \div \left(\frac{h}{\cos x}\right) = \frac{(R + r) \times \cos x}{h}$

$= \sin z$; and $x - z = m$.

The required angle n is obtained by a similar method. In this case $\frac{b}{h} = \tan y$; and $\frac{BQ}{OQ} = (R + r) \div \left(\frac{h}{\cos y}\right) = \frac{(R + r) \times \cos y}{h} =$ $\sin w$. Then $y - w = n$.

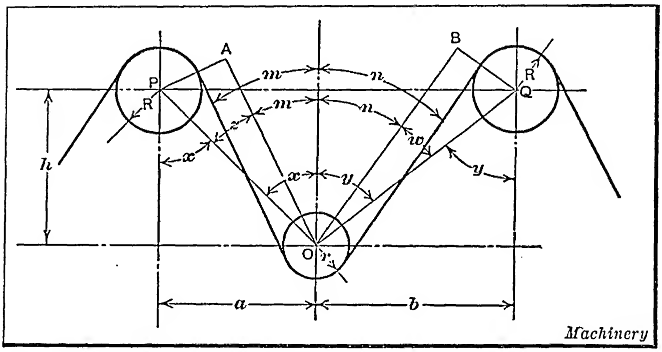

Fig. 8. Diagram showing Method of finding Angle of Special Screw Thread

Example: Let $a = 0.25$; $b = 0.375$; $h = 0.3125$; $R = 0.05$; $r = 0.04$. Then $\tan x = \frac{a}{h} = \frac{0.25}{0.3125} = 0.80000$; hence $x = 38$ degrees 39 minutes 35 seconds.

$$\text{Sin } z = \frac{AP}{OP} = \frac{(R + r) \times \cos x}{h} = \frac{0.09 \times 0.78087}{0.3125} = 0.22489;$$

hence $z = 12$ degrees 59 minutes 47 seconds.

Then $m = x - z = 38$ degrees 39 minutes 35 seconds $- 12$ degrees 59 minutes 47 seconds $= 25$ degrees 39 minutes 48 seconds.

$\text{Tan } y = \frac{b}{h} = \frac{0.375}{0.3125} = 1.20000$; hence $y = 50$ degrees 11 minutes 40 seconds.

$$\text{Sin } w = \frac{(R + r) \cos y}{h} = \frac{0.09 \times 0.64018}{0.3125} = 0.18437; \text{ hence } w =$$

10 degrees 37 minutes 28 seconds.

Then $n = y - w = 50$ degrees 11 minutes 40 seconds $- 10$ degrees 37 minutes 28 seconds $= 39$ degrees 34 minutes 12 seconds.

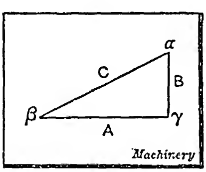

Fig. 9. To find Angles, having Differences between Sides

To Determine Angles of Right Triangle for Given Differences between Sides. Find the angles of a right triangle when the difference between the hypotenuse and the greater leg is equal to the difference between the two legs.

Solution: In Fig. 9, C is the hypotenuse, A is the greater leg, and B is the shorter leg. Then, $C - A = A - B$, from which, $C = 2A - B$. But $C^2 = A^2 + B^2$. Substituting the value of C previously found, $4A^2 - 4AB + B^2 = A^2 + B^2$; from which $B =$

¾A. Hence, when $A = 4$, $B = 3$, and $C = 2 \times 4 - 3 = 5$. Therefore, $\sin \beta = ⅗ = 0.6$, and $\beta = 36$ degrees, 52 minutes, 11.63 seconds. Angle $\alpha = 90$ degrees $- \beta = 90$ degrees $- 36$ degrees, 52 minutes, 11.63 seconds $= 53$ degrees, 7 minutes, 48.37 seconds. Any other value than 4 might have been substituted for A, and the result would have been the same.

To Find Angle ϕ, Having the Three Dimensions Shown in Fig. 10. In Fig. 10 the problem is to find angle ϕ which is required in laying

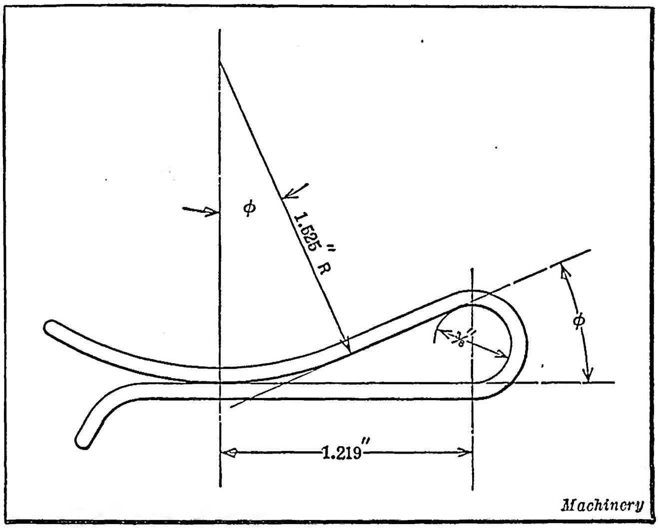

Fig. 10. To find Angle ϕ having the Three Dimensions given on Diagram

out a perforating, embossing, and cutting-off die for producing a spring clamp. The illustration shows a side view of the clamp.

Solution: First construct the diagram shown in Fig. 11, assuming that $c = 1.219$ inches, $R = 1.525$ inches, and $r = 3/16$ inch. From the right triangle AOB, it will be seen that

$$\operatorname{Tan} \alpha = \frac{R - r}{c} \quad (1) \qquad \text{and} \qquad h = \frac{R - r}{\sin \alpha} \quad (2)$$

From the right triangle AOE,

$$h = \frac{R + r}{\sin (\phi + \alpha)} \tag{3}$$

From (2) and (3) by comparison,

$$\operatorname{Sin} (\phi + \alpha) = \frac{R + r}{R - r} \sin \alpha \tag{4}$$

Inserting numerical values in (1), we find

$$\alpha = 47 \text{ degrees } 39 \text{ minutes } 14 \text{ seconds}$$

Then from (4) we find

$$\phi + \alpha = 71 \text{ degrees } 8 \text{ minutes } 23 \text{ seconds}$$

$$\phi = 71 \text{ deg. } 8 \text{ min. } 23 \text{ sec.} - 47 \text{ deg. } 39 \text{ min. } 14 \text{ sec.}$$
$$= 23 \text{ degrees } 29 \text{ minutes } 9 \text{ seconds.}$$

To Determine Angles of a Triangle when Lengths of Sides are Known. The problem is to calculate the angles of a triangular

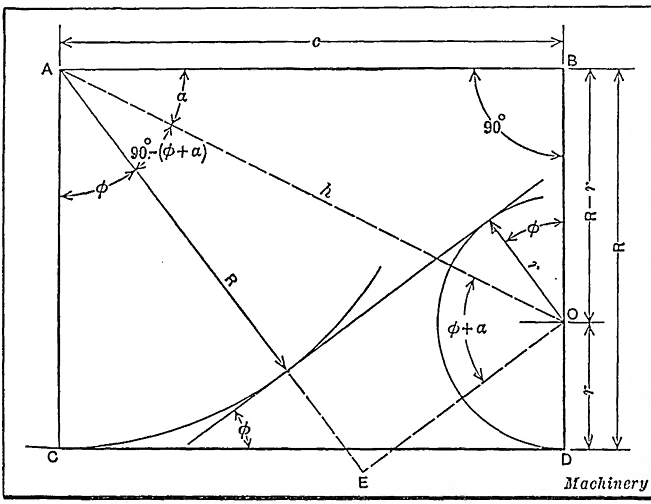

Fig. 11. Values used in solving the Problem represented by Fig. 10

piece, the lengths of the sides of which are $1\frac{5}{16}$ inches, $3\frac{1}{4}$ inches, and $3\frac{11}{16}$ inches.

Solution: There are a number of formulas for calculating the angles, one of which is given in MACHINERY'S HANDBOOK. This formula may be put into an easier form for calculating by a slight rearrangement and factoring of the terms, thus:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2b}\left[c + \frac{(b + a)(b - a)}{c}\right]$$

also

$$\cos B = \frac{1}{2a}\left[c - \frac{(b + a)(b - a)}{c}\right]$$

In these two formulas, a is the shortest side and c the longest side.

In Fig. 12, it will be noticed that the angles A, B, and C are opposite the sides a, b, and c, respectively. The chief advantage of these two formulas is that, having calculated the fraction by one formula, the result can be substituted in the second formula; also, it is not necessary to square three different numbers. In the present case, $a = 1\frac{5}{16} = 1.3125$, $b = 3\frac{1}{4} = 3.25$, and $c = 3\frac{11}{16} = 3.6875$; therefore,

$$\cos A = \frac{1}{2 \times 3.25}\left[3.6875 + \frac{(3.25 + 1.3125)(3.25 - 1.3125)}{3.6875}\right]$$
$$= \frac{1}{6.5}(3.6875 + 2.39725) = 0.93611.$$

The angle the cosine of which is 0.93611 is 20 degrees, 35 minutes, 30 seconds. Again substituting, $\cos B = \frac{1}{2 \times 1.3125}(3.6875 - 2.39725) = 0.49153 = \cos 60$ degrees, 33 minutes, 31 seconds. Angle $C = 180$ degrees $-$ (20 degrees, 35 minutes, 30 seconds $+$ 60 degrees, 33 minutes, 31 seconds) $=$ 98 degrees, 50 minutes, 59 seconds.

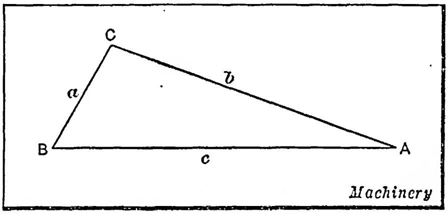

Fig. 12. To find Angles when Lengths of Sides are known

How to Find Angles A and B, Fig. 13. Find the angles A and B from the angle and dimensions given in Fig. 13.

Solution: From trigonometry we have the relations:

$$r^2 = b^2 + c^2 - 2bc \cos B \qquad (1)$$

$$r^2 = a^2 + c^2 - 2ac \cos A \qquad (2)$$

Subtracting (2) from (1)

$$b^2 - a^2 - 2bc \cos B + 2ac \cos A = 0 \qquad (3)$$
$$A = 15 \text{ degrees} - B$$

$$\text{Cos } A = \cos(15 \text{ deg.} - B) = \cos 15 \text{ deg.} \times \cos B + \sin 15 \text{ deg.} \times \sin B$$

Substituting this value for $\cos A$ in (3)

$$b^2 - a^2 - 2bc \cos B + 2ac(\cos 15 \text{ deg.} \times \cos B + \sin 15 \text{ deg.} \times \sin B) = 0$$

Expanding, combining, and arranging terms,

$$2c(a \cos 15 \text{ deg.} - b) \cos B + 2ac \sin 15 \text{ deg.} \sin B = a^2 - b^2 \qquad (4)$$

This is an equation of the form

$$m \cos B + n \sin B = q \qquad \text{(A)}$$

in which

$m = 2c\,(a \cos 15 \text{ degrees} - b)$
$n = 2ac \sin 15 \text{ degrees}$
$q = a^2 - b^2 = (a + b)\,(a - b)$

Solving this equation by the methods of trigonometry

$$\tan x = \frac{m}{n} \qquad \text{(B)}$$

$$\text{Sin}\,(x + B) = \frac{q \sin x}{m} \qquad \text{(C)}$$

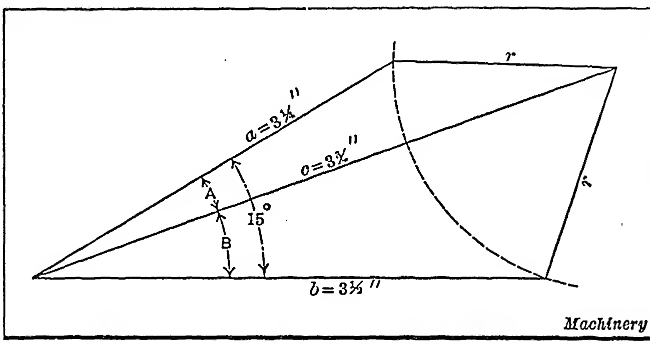

Fig. 13. To find Angles *A* and *B*

Inserting numerical values in (4) we find m, n, and q.

From (B)

$$x = 156 \text{ degrees } 47 \text{ minutes}$$

Then from (C)

$$(x + B) = 165 \text{ degrees } 46 \text{ minutes}$$

$B = 165$ deg. 46 min. $-$ 156 deg. 47 min. $=$ 8 deg. 59 min.
$A = 15$ deg. $- B = 15$ deg. $-$ 8 deg. 59 min. $=$ 6 deg. 1 min.

Another Solution: The following solution of this problem shows how the same result may be obtained by another method. First draw line XY, Fig. 14, connecting the intersections of the dotted circular arc with lines a and b. Then, from center O draw a line perpendicular to line XY, continuing until it meets line b at P. From trigonometry,

$$(XY)^2 = a^2 + b^2 - 2ab \cos 15 \text{ degrees}$$

Substituting the given values in this formula,

$$(XY)^2 = 10.5625 + 12.25 - 21.975 = 0.8378$$
$$XY = 0.9153 \text{ inches}$$

Then, according to the law of sines,

$$\frac{XY}{\sin 15 \text{ deg.}} = \frac{a}{\sin E} \qquad \text{Sin } E = \frac{a \sin 15 \text{ deg.}}{XY}$$

Substituting values,

$$\text{Sin } E = \frac{3.25 \times 0.25882}{0.9153} = 0.919$$

Therefore,

$$E = 66 \text{ degrees } 47 \text{ minutes}$$

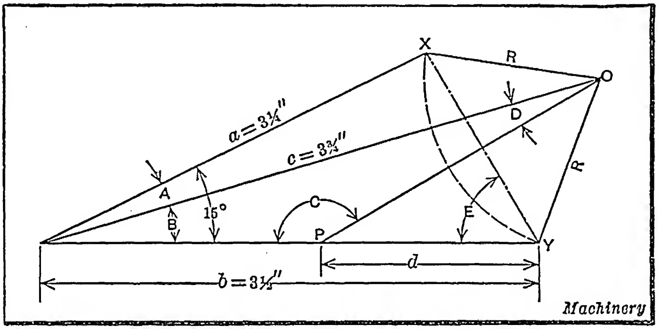

Fig. 14. Diagram for Second Method of finding Angles *A* and *B*

The triangle OXY is an isosceles triangle, and for this reason line OP bisects line XY. It is now possible to determine the length of line d by the formula

$$d = \frac{XY \sec E}{2}.$$

Substituting the known values and solving,

$$d = 1.1609 \text{ inches}$$

and

$$b - d = 3.5 - 1.1609 = 2.3391 \text{ inches}$$

It is evident that

$$C = E + 90 \text{ degrees}$$

Thus

$$C = 66 \text{ deg. } 47 \text{ min.} + 90 \text{ deg.} = 156 \text{ deg. } 47 \text{ min.}$$

Again, according to the law of sines

$$\frac{b-d}{\sin D} = \frac{c}{\sin C} \qquad \text{Sin } D = \frac{(b-d) \sin C}{c}$$

Inserting the proper values,

$$\text{Sin } D = \frac{2.3391 \times 0.39421}{3.75} = 0.24589$$

and

$$D = 14 \text{ degrees } 14 \text{ minutes}$$

$B = 180$ deg. $- (C + D) = 180$ deg. $-$ (156 deg. 47 min. $+$ 14 deg. 14 min.) $= 8$ deg. 59 min.

and

$$A = 15 \text{ deg.} - B = 15 \text{ deg.} - 8 \text{ deg. } 59 \text{ min.} = 6 \text{ deg. } 1 \text{ min.}$$

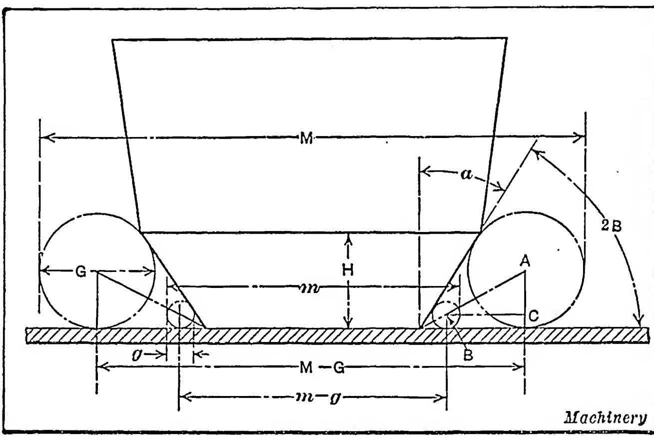

Fig. 15. Wire Method employed for checking Angle of Conical Surface

To Check Angle of Tapered Plug with Two Sizes of Wires. The angle of a tapered plug may be measured by taking two micrometer readings over two pairs of wires of different sizes. This method is useful for checking the angle of a short taper, the distance H, Fig. 15, being comparatively small.

Solution: In the diagram, G and g equal the diameters of the maximum and minimum wires, and M and m the micrometer readings over the maximum and minimum wires, respectively. Then

$M - G$ = distance between centers of maximum wires;
$m - g$ = distance between centers of minimum wires.

In the triangle ABC,

$$\tan B = \frac{AC}{BC}$$

But

$$AC = \frac{G - g}{2}$$

and

$$BC = \frac{(M - G) - (m - g)}{2} = \frac{(M - m) - (G - g)}{2}$$

Therefore, substituting in the first equation:

$$\tan B = \frac{\frac{G - g}{2}}{\frac{(M - m) - (G - g)}{2}} = \frac{G - g}{(M - m) - (G - g)}$$

$$a = 90 \text{ degrees} - 2B$$

and

$$2 \tan a = \text{the taper per inch}$$

Double and Compound Angles. Double or compound angles can be solved by the simple method which follows: Suppose that a 45-degree angle on elevation CAB (Fig. 16) is to be swung 30 degrees in a horizontal plane about point A. The plan view of this angle before being swung around would be a straight line DE equal to the length of the base AB. Now swing the line DE through an arc of 30 degrees to the position DF and draw a line perpendicular to line AB from the point F. Call this line FJ and lay off HJ equal to CB. Triangle HAJ is now the true elevation of the triangle in its new position DF, the side AC being represented by line AH, and side BC, by side JH. Now $AB = DE = DF$; and $BC = HJ$.

$BC = AB \tan 45$ degrees; therefore, $HJ = AB \tan 45$ degrees. Further, $DG = DF \cos 30$ degrees; and as $DG = AJ$, $AJ = DF \cos 30$ degrees $= AB \cos 30$ degrees.

$$\text{Tan } \alpha = \frac{HJ}{AJ} = \frac{AB \tan 45 \text{ degrees}}{AB \cos 30 \text{ degrees}} = \frac{\tan 45 \text{ degrees}}{\cos 30 \text{ degrees}}$$

This principle can be carried a step further and a compound angle worked out in the same manner, resolving the problem into a series of simple motions. In using this method any convenient side of the triangle may be taken as unity and the problem solved with the trigonometric functions of the angles. If a book giving the logarithms of the trigonometric functions is available, the problems are made comparatively simple and the chances of making errors are considerably reduced.

To Calculate Resultant Angles. Draftsmen and machine designers sometimes find it necessary or desirable to calculate what is

commonly termed "the resultant angle." This problem is frequently met with in making drawings for automobile and airplane parts. A clearer understanding of the problem, as it is encountered in practice, will be obtained by referring to Fig. 17. Suppose, for instance, that pin *A* is set at a given angle *B* with respect to the block *C*, as shown in the front elevation view. Now assume that pin *A* is also tilted to another given angle *D* in a different plane, as indicated in the side elevation. It is evident that the resultant angle *E* must be known by the workman in order to enable him to drill the hole for pin *A* at the correct angle in the plane of section *X*-*X*. First it is necessary to calculate angle *F* from the given angles *B* and *D*. When this angle has been determined, the draftsman can lay out the plan view quickly and with greater accuracy than would ordinarily be obtained by the projection method. It might be mentioned here that similar calculations are required in determining the position and the angle of inclination of the edge that is formed by two converging sides of a pyramid or wedge.

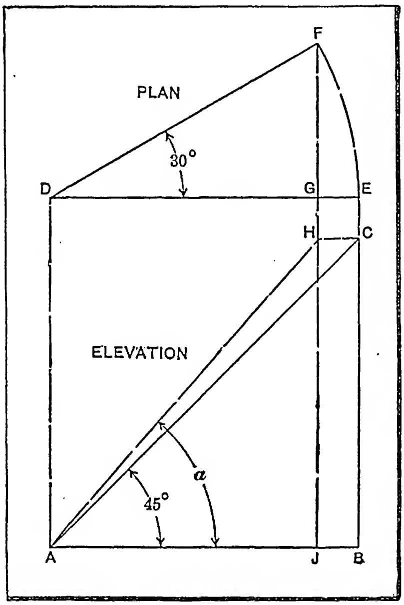

Fig. 16. Diagram illustrating Solution of Double and Compound Angles

In the illustration, let it be assumed that the front and side elevations showing the pin and block have been drawn and that the draftsman has also drawn the plan view of the block. In order to show the pin in the plan view, it is necessary to determine angle *F*, after which angle *E* may be readily calculated. The formulas for obtaining these angles are,

$$\text{Tan } F = \frac{\tan B}{\tan D}$$

and

$$\text{Tan } E = \sqrt{\tan^2 B + \tan^2 D}$$

As an example showing the use of these formulas let it be assumed that angle $B = 7$ degrees and angle $D = 10$ degrees.

Then

$$\text{Tan } F = \frac{\tan\ 7 \text{ degrees}}{\tan 10 \text{ degrees}} = \frac{0.12278}{0.17633} = 0.69631$$

Thus

$$F = 34 \text{ degrees } 51 \text{ minutes}$$

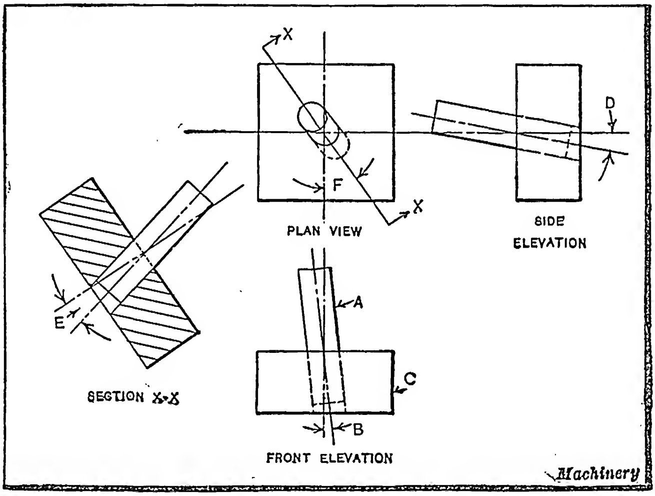

Fig. 17. Diagrams for illustrating Resultant Angle Problems

Now

$$\begin{aligned} \text{Tan } E &= \sqrt{\tan^2 7 \text{ degrees} + \tan^2 10 \text{ degrees}} \\ &= \sqrt{0.12278^2 + 0.17633^2} = 0.21487 \end{aligned}$$

Therefore

$$E = 12 \text{ degrees } 8 \text{ minutes}$$

To Express Angles in Circular Measure. In geometry, the measuring unit for angles is the right angle; in practical trigonometry, angles are measured in degrees, minutes, and seconds; in what is called analytical trigonometry, the measuring unit is the radian. The first two of these measures are concerned only with the angle itself or with a part of a revolution; as a consequence,

no linear unit can be used in comparing one angle with another, and both measures are called angular measures. In certain formulas and in mathematical investigations, it is advisable to have a linear measure for comparing or denoting angles, and in such cases the arc is used instead of the angle. Now, a semicircle of a radius r has a length equal to πr; but since this line is rather long to use as a unit of arc measurements, it is customary to use that part of the semicircle which is equal in length to the radius. It will readily be perceived that the angle which this arc subtends is constant; in other words, the angle will be the same whatever the radius. The value of this angle in angular measure is readily found. Thus, since the semicircumference subtends two right angles, or 180 degrees, and its length is πr, we have the proportion $\frac{180}{\pi r} = \frac{x}{r}$, from which $x = 57.29577951$ degrees $= 57$ degrees, 17 minutes, 44.8 seconds, very nearly. Hence, if an angle is given in degrees, minutes, and seconds and it is desired to find its value in radians, reduce the minutes and seconds to a decimal of a degree, and divide the angle by 57.29578 or multiply it by $\frac{\pi}{180} = 0.0174532925 +$. As an example showing the use of circular measure in a formula, the area of a sector is equal to the product of one-half the arc by the radius; if the central angle is V radians, the length of the arc is rV; the area of the sector is $A = \frac{1}{2}rV \times r = \frac{1}{2}r^2V$. By substituting $v = \frac{1}{2}V$, $A = r^2v$, a very convenient and simple form. When angular measure is used, $A = 0.0174532925 \times r^2v$, when v is in degrees.

To Interpolate to Seconds. Suppose the natural tangent is 0.63154; how is the angle in degrees, minutes and seconds determined, using the table in MACHINERY'S HANDBOOK?

Method: Referring to MACHINERY'S HANDBOOK, the angle whose tangent is 0.63154 is seen to lie between 32 degrees, 16 minutes (the tangent of which is 0.63136) and 32 degrees, 17 minutes (the tangent of which is 0.63177). The difference between these two values is 0.00041; the difference between the tangent of the smaller angle and the given tangent is 0.00018; and the difference between the two angles in the table is 1 minute = 60 seconds. Assume that the following proportion is true, as it will be for practical purposes, $x : 60 = 0.00018 : 0.00041$, from which $x = \frac{60 \times 0.00018}{0.00041}$ $= 26$ seconds; hence the angle whose tangent is 0.63154 equals 32 degrees, 16 minutes, 26 seconds. Therefore, the rule is: Find two consecutive functions in the table one of which is greater and

the other less than the given function, and take their difference. Find the difference between the given function and the function corresponding to the smaller of the two angles; multiply this difference by 60 and divide the product by the difference previously found. The quotient will be the number of seconds to be added to the smaller angle in order to obtain the angle required.

To illustrate the application of the rule by another example, find the angle equivalent to the sine 0.44633. The table shows that this angle is between 26 degrees 30 minutes and 26 degrees 31 minutes, as the sine of the former angle is 0.44620 and of the latter, 0.44646. The difference between these two functions equals $0.44646 - 0.44620 = 0.00026$. The difference between the given function and the function corresponding to the smaller angle, equals $0.44633 - 0.44620 = 0.00013$. Hence the number of seconds to be added to the smaller angle equals $\frac{60 \times 0.00013}{0.00026} = 30$ seconds; therefore the required angle equals 26 degrees 30 minutes 30 seconds.

CHAPTER IV

SOLUTION OF TRIANGLES TO DETERMINE LINEAR DIMENSIONS

The problems in this chapter illustrate how linear dimensions, such as the lengths of the sides of triangles, are determined when certain related dimensions and angles are given. The simple diagrams used to show the given dimensions and the unknown values may represent the center-to-center distances of shafting or gearing, the sides of templets, the edges of cutting tools, or numerous other practical applications. These problems are typical examples of applied geometry and trigonometry, as obtained from drafting-room practice.

Law of Sines and Cosines. In a triangle, any side is to any other side as the sine of the angle opposite the first side is to the sine of the angle opposite the other side; or, if a and b be the sides, and A and B the angles opposite them:

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

In a triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice their product times the cosine of the included angle; or if a, b, and c be the sides and the angle opposite side a be denoted A, then:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

and

$$a = \sqrt{b^2 + c^2 - 2bc \cos A}$$

These two laws, together with the proposition that the sum of the three angles equals 180 degrees, are the basis of all formulas relating to the solution of triangles.

To Find Lengths of Sides b and c, Fig. 1, when Side a and Sum of the Other Sides are Known. In Fig. 1 ABC is a right triangle, right-angled at C. If the side a and the sum of the other two sides are known, how can the lengths of c and b be found? Also, if c and the sum of a and b are known, how can the lengths of a and b be found?

Solution: For the first case, let $s = c + b$, then $b = s - c$. But $c^2 = a^2 + b^2$. Substituting the value of b,

$$c^2 = a^2 + s^2 - 2cs + c^2$$

or,

$$2cs = s^2 + a^2$$

Therefore,

$$c = \frac{s^2 + a^2}{2s}$$

For example, suppose $c + b = 20$ and $a = 5$, then

$$c = \frac{20^2 + 5^2}{2 \times 20} = 10.625$$

and

$$b = 20 - 10.625 = 9.375$$

For the second case, let $a + b = s$, then $a = s - b$. But $c^2 = a^2 + b^2$. Substituting the value of a, $c^2 = s^2 - 2sb + b^2 + b^2$.

Transposing, combining, and reducing,

$$b^2 - sb = \frac{c^2 - s^2}{2}$$

whence,

$$b = \tfrac{1}{2}\left(s + \sqrt{2c^2 - s^2}\right)$$

If the + sign is used, the length of the longer side will be obtained; and if the − sign is used, the length of the shorter side will be found. For example, if

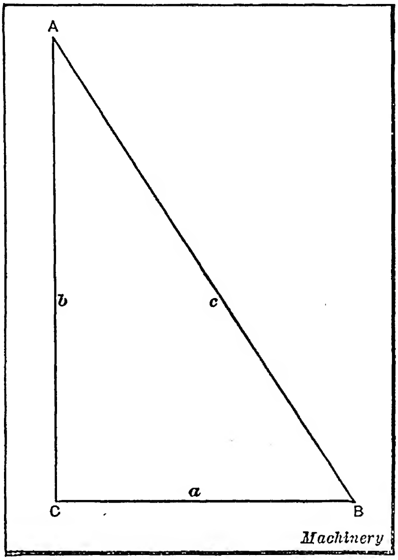

Fig. 1. To find Sides b and c when Side a and the Sum of the Other Sides are known

$$c = 10\tfrac{5}{8} \text{ and } a + b = 14\tfrac{3}{8}$$

$$b = \tfrac{1}{2}\left(14\tfrac{3}{8} \pm \sqrt{2 \times 10.625^2 - 14.375^2}\right)$$

$$= \tfrac{1}{2}(14.375 \pm 4.375) = 9.375 \text{ or } 5$$

That is, the longer side is 9.375 and the shorter is 5.

To Find Dimensions x and y, Fig. 2, when Three Sides are Known. A problem quite often encountered in lay-out work is illustrated in Fig. 2. It is required to find the values of x and y, these lengths being measured from the intersection of the perpendicular h with the base c. The three sides of the triangle are the only known values.

Solution: The method that might first suggest itself is to find the angle A (or B) by some such formula as

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1}$$

and then solve the right triangle for y by the formula

$$y = b \cos A \tag{2}$$

Formulas (1) and (2) can be combined as follows

$$y = \frac{b^2 + c^2 - a^2}{2c} \tag{3}$$

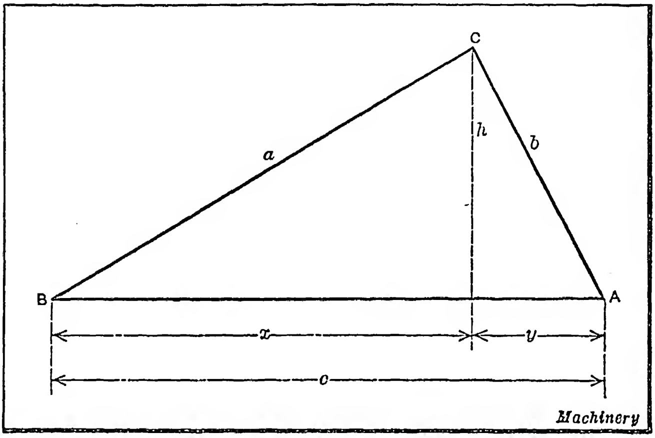

Fig. 2. To find Dimensions x and y when Three Sides are known

The value of x can be determined in a similar manner.

The solution of a formula of this type is simple enough when a, b, and c are values having but two or three digits, or if they are simple integers, but if they contain four or five figures each, the work becomes laborious and mistakes are likely to occur when squaring these numbers.

A convenient method to use in the solution of this problem involves the following geometrical proposition: In any oblique triangle where the three sides are known, the ratio of the length of the base to the sum of the other two sides equals the ratio of the difference between the length of the two sides to the difference between the lengths x and y.

Therefore, if in Fig. 2, $a = 14$, $b = 12$ and $c = 16$ inches, then

$$c : (a + b) = (a - b) : (x - y)$$
$$16 : 26 = 2 : (x - y)$$
$$(x - y) = \frac{26 \times 2}{16} = 3\tfrac{1}{4} \text{ inches}$$

But

$$x = \frac{(x + y) + (x - y)}{2} = \frac{16 + 3\tfrac{1}{4}}{2} = 9.625 \text{ inches}$$

and

$$y = \frac{(x + y) - (x - y)}{2} = \frac{16 - 3\tfrac{1}{4}}{2} = 6.375 \text{ inches}$$

To Determine Altitude of an Obtuse-angle Triangle. Find the distances x and y from the dimensions given in Fig. 3.

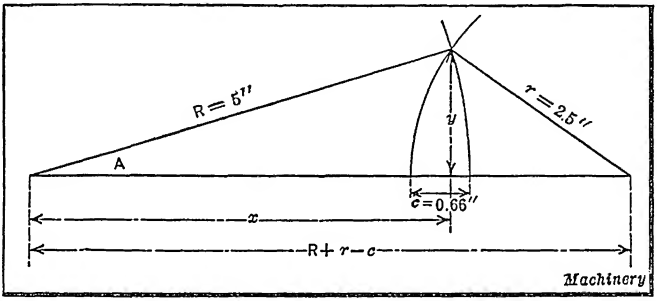

Fig. 3. To find Dimensions x and y having Dimensions given on Diagram

First Solution: The following method of solving the problem stated requires only an elementary knowledge of trigonometry and the use of a table of trigonometrical functions. In Fig. 3,

$$R + r - c = 5 + 2.5 - 0.66 = 6.84 \text{ inches}$$

According to trigonometry

$$\text{Cos } A = \frac{R^2 + (R + r - c)^2 - r^2}{2R\,(R + r - c)}$$

Substituting the known values and solving,

$$\text{Cos } A = \frac{5^2 + 6.84^2 - 2.5^2}{2 \times 5 \times 6.84} = \frac{25 + 46.7856 - 6.25}{2 \times 5 \times 6.84}$$

$$\text{Cos } A = 0.9581228 \quad \text{and} \quad A = 16 \text{ deg. } 38 \text{ min. } 25 \text{ seconds}$$
$$\text{Sin } A = 0.28636$$
$$x = R \times \cos A = 5 \times 0.9581228 = 4.790614 \text{ inches}$$
$$y = R \times \sin A = 5 \times 0.28636 = 1.4318 \text{ inches}$$

Second Solution: In Fig. 4 we have:

$$x + N = R + r - C = 5 + 2.5 - 0.66 = 6.84 \text{ inches}$$

Also the sum of the two shorter sides is:

$$R + r = 5 + 2.5 = 7.5 \text{ inches}$$

and the difference between the two shorter sides is:

$$R - r = 5 - 2.5 = 2.5 \text{ inches}$$

Then

$$x + N : R + r :: R - r : x - N$$

or

$$6.84 : 7.5 :: 2.5 : x - N$$

Therefore

$$x - N = \frac{7.5 \times 2.5}{6.84} = 2.74123 \text{ inches}$$

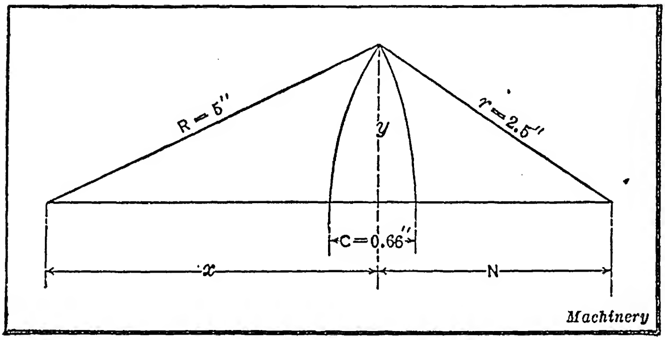

Fig. 4. Diagram for Second Solution of Problem represented by Fig. 3

According to arithmetic, if the sum of two numbers and their difference be given, the greater of the two numbers is equal to one-half the sum of their sum and difference.

$$x = \frac{(x + N) + (x - N)}{2} = \frac{6.84 + 2.74123}{2} = 4.79062 \text{ inches}$$

$$y = \sqrt{5^2 - 4.79062^2} = 1.43176, \text{ or } 1.4318 \text{ inches}$$

Third Solution: This solution involves the application of a simple formula that is seldom given in handbooks, but which is useful in solving problems involving oblique triangles. From Fig. 5, the formula is as follows:

$$D = \frac{R^2 - r^2}{2A}$$

This formula is derived from the following equations:

$$\sqrt{R^2 - y^2} = \frac{A}{2} + D \qquad (1)$$

$$\sqrt{r^2 - y^2} = \frac{A}{2} - D \qquad (2)$$

Squaring both sides we have,

$$4R^2 - 4y^2 = A^2 + 4AD + 4D^2 \qquad (1)$$

$$4r^2 - 4y^2 = A^2 - 4AD + 4D^2 \qquad (2)$$

Subtracting (2) from (1) we have,

$$4R^2 - 4r^2 = 8AD \text{ or}$$

$$R^2 - r^2 = 2AD$$

$$D = \frac{R^2 - r^2}{2A}$$

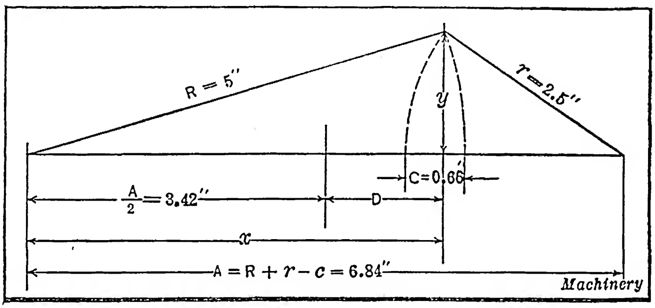

Fig. 5. Diagram for Third Solution of Problem represented by Fig. 3

Substituting the numerical values of the given problem,

$$D = \frac{25 - 6.25}{13.68} = 1.37061 \text{ inches}$$

$$x = 3.42 + 1.37061 = 4.79061 \text{ inches}$$

$$y = \sqrt{R^2 - x^2} = \sqrt{25 - 22.9441} = 1.4318 \text{ inches}$$

Formulas for Calculating Sides of Trapezoid. The problem is to find the fourth side of a trapezoid when three sides and the altitude are given. From the diagram, Fig. 6, it will be evident that there are four cases to be considered in the solution of this problem. The formulas for each case are given in the following:

1. Given sides a, c, and d:

$$b = d - (\sqrt{a^2 - h^2} + \sqrt{c^2 - h^2})$$

2. Given sides a, b, and c:

$$d = b + (\sqrt{a^2 - h^2} + \sqrt{c^2 - h^2})$$

3. Given sides b, c, and d:

$$a = \sqrt{(d - b - \sqrt{c^2 - h^2})^2 + h^2}$$

4. Given sides a, b, and d:

$$c = \sqrt{(d - b - \sqrt{a^2 - h^2})^2 + h^2}$$

The derivation of these equations is as follows: Draw lines x and y parallel to altitude h from the points B and C, respectively. Then x forms one leg of the right triangle ABE and y forms one leg of the right triangle DCF. Then (by construction) x, y, and h are equal. Now in any right triangle the hypotenuse squared

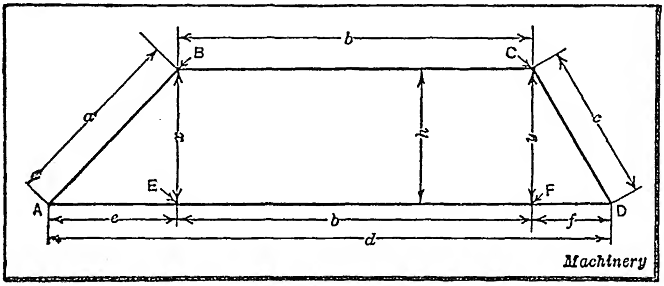

Fig. 6. To find the Fourth Side of a Trapezoid when Three Sides and the Altitude are given

equals the base squared plus the altitude squared. Therefore we have the formulas:

$$e = \sqrt{a^2 - h^2} \quad \text{and} \quad f = \sqrt{c^2 - h^2} \qquad (1)$$

$$e = d - (b + f) \quad \text{and} \quad f = d - (b + e) \qquad (2)$$

$$a = \sqrt{e^2 + h^2} \quad \text{and} \quad c = \sqrt{f^2 + h^2} \qquad (3)$$

Then by substituting these values in their proper places according to the accompanying diagram, we obtain the four formulas given in the preceding.

To Determine the Lengths of Two Sides of an Oblique Triangle. Fig. 7 shows an oblique triangle in which the difference between the lengths of sides b and c = 1 inch; the length of side a = 13 inches; and angle A = 120 degrees. How can the lengths of sides b and c be found?

First Solution: From the accompanying illustration, Fig. 7,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Now as $\cos A = -\frac{1}{2}$, we have $a^2 = b^2 + c^2 + bc$

According to the conditions of the problem, $b = c + 1$. Therefore

$$a^2 = c^2 + 2c + 1 + c^2 + c^2 + c = 169$$

Then

$$c^2 + c = 56 \qquad \text{and} \qquad (c + \tfrac{1}{2})^2 = 56\tfrac{1}{4} \qquad \text{or} \qquad 225 \div 4$$

and

$$c + \tfrac{1}{2} = \frac{15}{2};\ c = 7 \text{ inches; and } b = 7 + 1 = 8 \text{ inches}$$

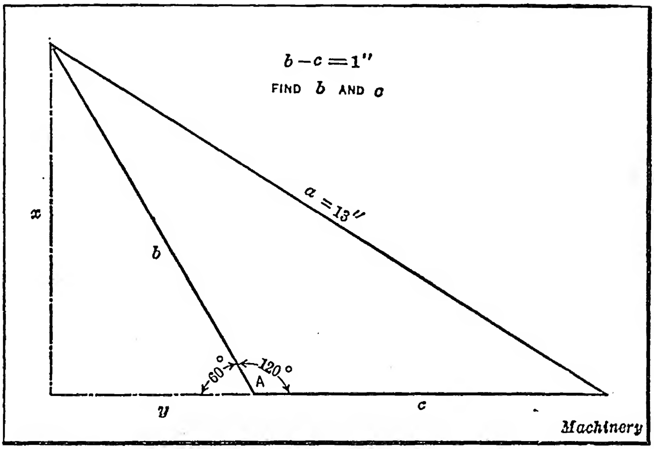

Fig. 7. To find Sides b and c, having the Difference between these Sides, the Length of Side a, and the Angle A

Second Solution: Referring to Fig. 7, $x^2 + y^2 = b^2$; $y = \frac{1}{2}b$; and $b = c + 1$.

Further

$$a^2 = x^2 + (c + y)^2 = x^2 + c^2 + 2cy + y^2$$

and

$$a^2 = b^2 + c^2 + 2cy = c^2 + b^2 + cb$$

Then by substitution

$$a^2 = c^2 + c^2 + 2c + 1 + c^2 + c = 3c^2 + 3c + 1 = 169$$

Thus

$$c^2 + c = 56 \quad \text{and} \quad c^2 + c + \tfrac{1}{4} = 56\tfrac{1}{4} = 225 \div 4$$

and

$$c + \tfrac{1}{2} = \frac{15}{2};\ c = 7 \text{ inches; } b = 7 + 1 = 8 \text{ inches}$$

Third Solution: In Fig. 8 continue QS to P and draw TP at right angles to PQ. It will be obvious that angle $TSP = 60$ degrees. Then, $PS = \frac{1}{2}\,ST$ and $PT = \sqrt{3} \times \frac{1}{2}\,ST$. Also, if $b - c = 1$, $b = c + 1$. Therefore,

$$QP = c + \frac{c+1}{2} \qquad \text{and} \qquad PT = \sqrt{3} \times \frac{c+1}{2}$$

Hence

$$\left(c + \frac{c+1}{2}\right)^2 + \left(\sqrt{3} \times \frac{c+1}{2}\right)^2 = 13^2 = 169$$

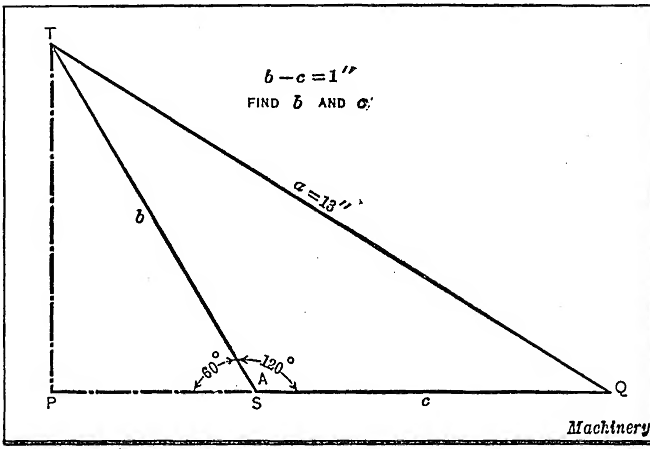

Fig. 8. Diagram for Another Solution of Problem represented by Fig. 7

$$\frac{9c^2 + 6c + 1}{4} + \frac{3c^2 + 6c + 3}{4} = \frac{12c^2 + 12c + 4}{4} = 169$$

$$3c^2 + 3c + 1 = 169$$
$$3c^2 + 3c = 168$$
$$c^2 + c = 56$$

Then

$$c^2 + c + 0.25 = 56.25$$
$$c + 0.5 = 7.5$$

and

$$c = 7.5 - 0.5 = 7 \text{ inches}$$

Finally, as

$$b = c + 1$$
$$b = 7 + 1 = 8 \text{ inches}$$

To Find the Lengths of Sides a and b, Fig. 9. As shown by the illustration, lines $a + b = 150$ inches, $o = 30$ inches and $c = 50$ inches. How can the length of sides a and b be determined?

Solution: The easiest method of solving this problem is to assume approximately correct values for lines a and b, then determine the magnitude of angle C for the assumed value of line a, and finally determine the length of line c with the assumed values of lines a and b. After a few trial calculations are made in the manner suggested, the exact value of line a can be obtained by interpolation, as explained in the following.

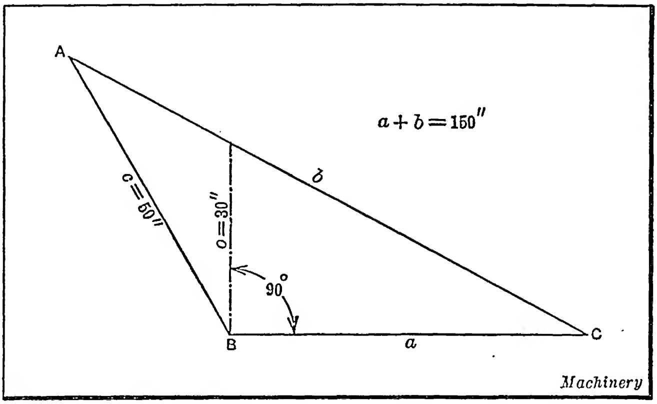

Fig. 9. To find Lengths of Sides a and b, having the Sum of these Sides and the Additional Dimensions given on Diagram

If the diagram illustrated is laid out to scale, it will be found that the length of line a is approximately 57 inches, and that of line b, 93 inches. Assuming these values to be correct, it is obvious that $\tan C = \frac{30}{57}$ and $C = 27$ degrees 45 minutes 31 seconds. Then, the length of line c can be calculated by means of the well-known trigonometrical formula for finding the length of a side of an oblique-angled triangle, when two sides and the included angle are known. Thus,

$$c = \sqrt{a^2 + b^2 - 2ab \cos C}$$

By inserting the assumed values in this formula, and solving, the length of line c, according to this calculation, will be 50.1607 inches. Thus, the error from the true value equals 50.1607 − 50 or 0.1607 inch.

Now, assuming line a to be 57.1 inches long and the length of line b to be 92.9 inches, angle C will be 27 degrees 43 minutes 2 seconds. Again calculating side c by means of the formula, it will be found to equal 49.9900 inches. In this case there is a minus error equal to 50 − 49.9900 or − 0.0100 inch. The difference between the two errors equals 0.1607 − (− 0.0100) or 0.1707 inch. The length of side a that will give an error of zero when calculating

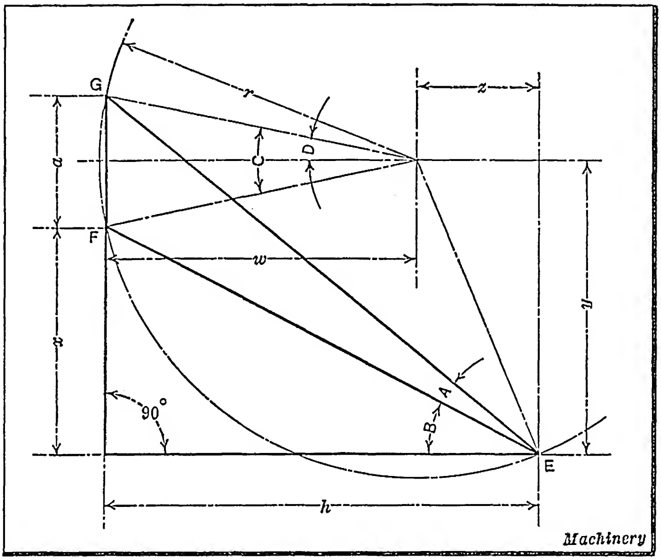

Fig. 10. To find Dimension x and Angle B, having Dimensions a, h, and Angle A

side c by means of the formula, can be determined by interpolating as follows:

$$57 + \frac{1607}{1707} \times 0.1 = 57.094 \text{ inches}$$

This gives the correct value of line a to the third decimal place. The value of line b equals 150 − 57.094 or 92.906 inches. If greater accuracy is required, repeat the computation with the last value found for side a and interpolate again.

To Find Dimension x and Angle B, Fig. 10. In the illustration $a = 1\frac{1}{4}$ inches, $h = 4$ inches, and angle $A = 12$ degrees. Find distance x and angle B.

Solution: Draw an arc through points E, F, and G, as shown, with r as a radius. According to a well-known theorem of geometry, if an angle at the circumference of a circle, between two chords, is subtended by the same arc as the angle at the center, between two radii, then the angle at the circumference is equal to one-half the angle at the center. This being true, angle C is twice the magnitude of angle A, and angle D = angle A = 12 degrees. It will now be readily observed that

$$r = \frac{a}{2 \sin D} = \frac{1.25}{2 \times 0.20791} = 3.0061$$

$$w = \frac{a}{2} \cot D = 0.625 \times 4.7046 = 2.9404$$

and

$$z = h - w = 4 - 2.9404 = 1.0596$$

Now

$$y = \sqrt{r^2 - z^2} = \sqrt{7.91388505} = 2.8131$$

and

$$x = y - \frac{a}{2} = 2.8131 - 0.625 = 2.1881 \text{ inches}$$

Finally,

$$\text{Tan } B = \frac{x}{h} = \frac{2.1881}{4} = 0.54703$$

and

$$B = 28 \text{ degrees } 40 \text{ minutes } 47 \text{ seconds}$$

Another Solution: In order to illustrate how the solution of a problem may be varied in many cases, the problem just dealt with will again be solved, using algebra and trigonometry. According to trigonometry, and with reference to Fig. 11,

$$x = h \tan B \tag{1}$$

and

$$\text{Tan } (A + B) = \frac{a + x}{h} \tag{2}$$

According to one of the important trigonometric formulas given in MACHINERY'S HANDBOOK:

$$\text{Tan } (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Substituting this value for tan $(A + B)$, Equation (2) becomes

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{a + x}{h} \tag{3}$$

Now substituting the value of x from Equation (1), Equation (3) becomes,

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{a + h \tan B}{h}$$

Clearing of fractions,

$$h(\tan A + \tan B) = (a + h \tan B)(1 - \tan A \tan B)$$

Expanding, combining like terms, and factoring,

$$\text{Tan } A\,(h \tan^2 B + a \tan B) = \tan A \left(\frac{a}{\tan A} - h\right)$$

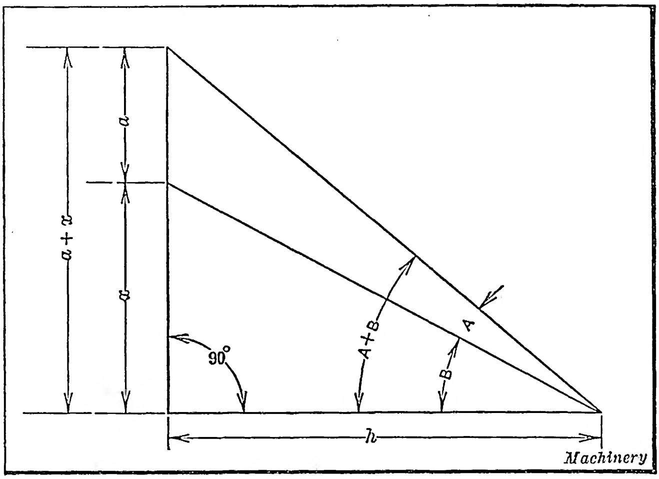

Fig. 11. Diagram for Second Solution of Problem represented by Fig. 10

Eliminating the factor tan A, substituting cot A for $\frac{1}{\tan A}$ and arranging terms,

$$h \tan^2 B + a \tan B - (a \cot A - h) = 0$$

Solving this quadratic equation for tan B,

$$\text{Tan } B = \frac{-a \pm \sqrt{a^2 - 4h(h - a \cot A)}}{2h}$$

Taking this equation with the plus sign before the radical and simplifying the terms under the radical,

$$\text{Tan } B = \frac{\sqrt{(a + 2h)(a - 2h) + 4ah \cot A} - a}{2h} \tag{4}$$

Inserting numerical values in Equation (4),

$$\text{Tan } B = \frac{\sqrt{94.0926 - 62.4375} - 1.25}{8}$$

$$= \frac{4.376286}{8} = 0.54703$$

and

$$B = 28 \text{ degrees } 40 \text{ minutes } 47 \text{ seconds}$$

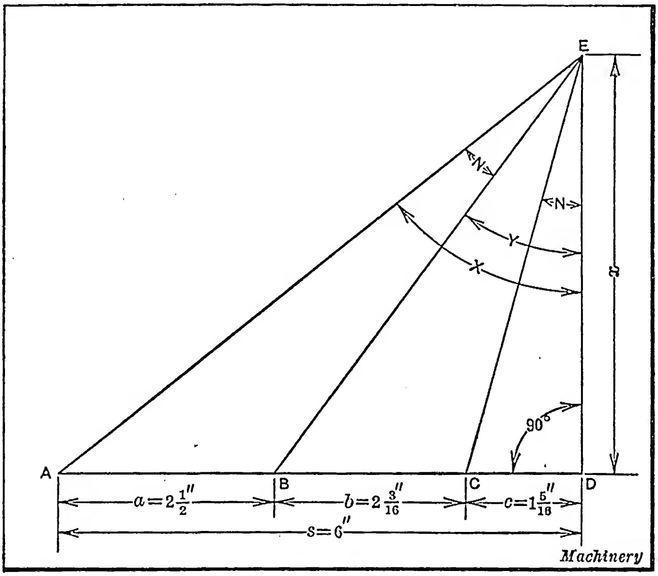

Fig. 12. To find Dimension x, having Sides a, b, and c, and Angle N

Now inserting the known values in Equation (1),

$$x = 4 \times 0.54703 = 2.1881 \text{ inches}$$

To Find Dimension x, Fig. 12, when Distances a, b, c and Angle N are given. Find dimension x in the illustration, the lengths of sides a, b, and c being given and angles AEB and CED being equal.

Solution: First, designate both angles AEB and CED as angle N. It is obvious that $N = X - Y$. Then,

$$\text{Tan } N = \tan (X - Y) = \frac{c}{x}$$

$$\text{Tan } X = \frac{s}{x} \qquad \text{and} \qquad \tan Y = \frac{b + c}{x}$$

From trigonometry the following relation is obtained:

$$\text{Tan }(X - Y) = \frac{\tan X - \tan Y}{1 + \tan X \tan Y}$$

Substituting the values previously found,

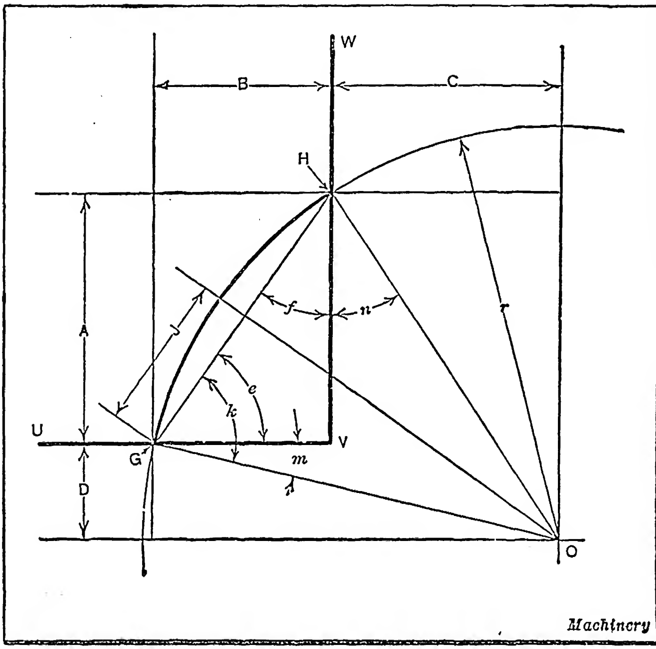

Fig. 13. To find Dimensions D and C, having A, B and Radius r

$$\frac{c}{x} = \frac{\dfrac{s - b - c}{x}}{1 + \dfrac{s\,(b + c)}{x^2}} = \frac{x\,(s - b - c)}{x^2 + s\,(b + c)}$$

Clearing of fractions and combining terms,

$$x^2\,(s - 2c - b) = cs\,(b + c)$$

$$x = \sqrt{\frac{cs\,(b + c)}{s - 2c - b}}$$

Reducing the dimensions of sides a, b, c, and s to sixteenths of an inch and inserting these numerical values,

$$x = \sqrt{\frac{21^2}{19}}$$

Therefore,

$$x = \frac{21\sqrt{19}}{19} = 4.8177 \text{ inches}$$

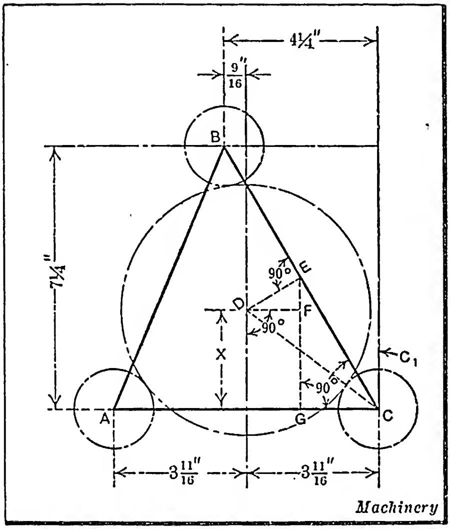

Fig. 14. To find Center Distances DC and X, having Dimensions given on Diagram

To Find Dimensions D and C, Fig. 13, Having A, B and Radius r. In the problem we have an arc of radius r, which cuts the corner of a right angle UVW in such a manner that the vertical line WV is cut at point H, and the horizontal line UV at point G, dimensions A and B being known. In order to determine the location of center O with respect to the sides of right angle UVW, it is necessary to find dimensions D and C.

The given dimensions are as follows:

A = amount cut off by arc on vertical line WV;
B = amount cut off by arc on the horizontal line UV; and
r = radius of arc.

With these dimensions given it is desired to find the horizontal distance C from point H to the center line passing through point O, and the vertical distance D from point G to the horizontal center line passing through point O.

Solution: By trigonometry

$$\text{Tan } e = \frac{A}{B} \qquad \text{and} \qquad \cot f = \frac{A}{B}$$

and by geometry

$$e + f = 90 \text{ degrees} \qquad \text{and} \qquad J = \tfrac{1}{2}\sqrt{A^2 + B^2}$$

Now

$$\cos k = \frac{J}{r} \qquad m = k - e \qquad \text{and} \qquad n = k - f$$

Then

$$D = r \sin m \qquad \text{and} \qquad C = r \sin n$$

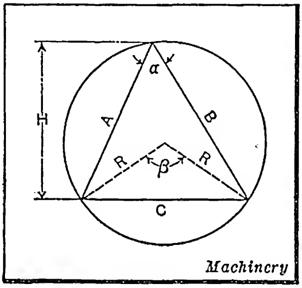

Fig. 15. Diagram illustrating Problem in Geometry

To Calculate Center Distances of Gearing Arranged as in Fig. 14. Three pinions of equal diameter are to be driven by one gear, and the center distances *DC* and *X* are required.

Solution: In any triangle, the product of two sides is equal to the product of the diameter of the circumscribed circle and the altitude upon the third side. Thus:

$$BC = \sqrt{4.25^2 + 7.25^2} = 8.4038 \text{ inches}$$
$$AB = \sqrt{3.125^2 + 7.25^2} = 7.8948 \text{ inches}$$

Diameter of circumscribed circle $= \dfrac{8.4038 \times 7.8948}{7.25} = 9.1512$ inches

$$DC = \frac{9.1512}{2} = 4.5756 \text{ inches}$$
$$X = \sqrt{4.5756^2 - 3.6875^2} = 2.709$$

This problem may also be solved as follows:

If the point of intersection of the vertical line through *C* and the horizontal line through *B* is denoted by *H*, as the point *E* bisects the line *BC*, the line *EG* is one-half of *HC*, or equal to $3\frac{5}{8}$ inches; and *GC* is one-half of *BH*, or equal to $2\frac{1}{8}$ inches. The triangles *DEF* and *CBH* are similar and their corresponding sides are proportional. Hence

$$\frac{EF}{DF} = \frac{BH}{CH} \qquad \text{or} \qquad EF = \frac{BH \times DF}{CH}$$

Now, $DF = BH - GC - \frac{9}{16}$; or $DF = 4\frac{1}{4} - 2\frac{1}{8} - \frac{9}{16} = 1\frac{9}{16}$. Therefore,

$$EF = \frac{4\frac{1}{4} \times 1\frac{9}{16}}{7\frac{1}{4}} = 0.916$$

whence

$$X = 3\tfrac{5}{8} - 0.916 = 2.709$$

$$DC = \sqrt{2.709^2 + 3.6875^2} = 4.5756 \text{ inches}$$

Still another solution will be given to illustrate different ways of analyzing the same problem: A line joining point D with apex B is exactly the same length as the line DC; therefore, $\tfrac{9}{16}^2 + (7\tfrac{1}{4} - x)^2 = x^2 + 3\tfrac{11}{16}^2$. Expanding, cancelling, transposing and reducing gives $x = 2.709$. Then $DC = \sqrt{2.709^2 + 3.6875^2} = 4.5756$ inches.

Proof that Product of AB Equals Diameter of Circle $\times H$, in Fig. 15. Prove that in any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle and the altitude upon the third side.

Solution: In Fig. 15 it is desired to prove that AB = diameter $\times H$.

The area of the triangle $= \dfrac{AB \sin \alpha}{2}$

Therefore, we will prove that

$$\frac{AB \sin \alpha}{2} = \frac{DH \sin \alpha}{2} = RH \sin \alpha$$

If an angle at the circumference of a circle, between two chords, is subtended by the same arc as the angle at the center, between the two radii, then the angle at the center equals two times the angle at the circumference, so $\beta = 2\alpha$.

The area of a triangle equals one-half the base times the altitude.

$$\tfrac{1}{2}C = R \sin \alpha$$

$$\text{Area} = R \sin \alpha H$$

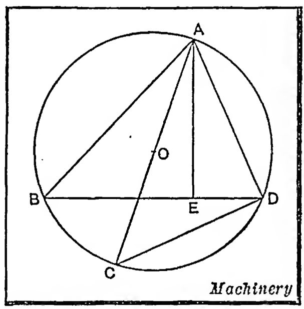

Fig. 16. Diagram for Second Analysis of Geometry Problem

Thus: $$\frac{AB \sin \alpha}{2} = RH \sin \alpha$$

Another analysis of this problem follows: Let ABD, Fig. 16, be the triangle and AE the altitude; draw AC through center O of the circumscribed circle and connect the points C and D. Then AEB is a right angle by construction, and ADC is a right angle by being measured by an arc of 180 degrees. Also, angles ABD and ACD are similar because they are measured by the same arc AD;

therefore, the right triangles ABE and ACD are similar. As a result, $AB : AC = AE : AD$; or $AB \times AD = AC \times AE$.

To Calculate the Center Distances a, b, c, and d, Fig. 17. From the dimensions given in the illustration, determine the horizontal and vertical distances between the centers of the circles shown.

Solution: Draw NB parallel to GA; join N and O; draw OC and OD perpendicular to NB and NM, respectively; draw MA perpendicular to GA. Angle $DOC = MNB$, and $\sin MNB = \frac{MB}{MN} = \frac{3/8 - 1/4}{2} = 1/16 = 0.0625$; therefore angle MNB = 3 degrees 35 minutes. In the right triangle NOC, $\cos NOC = \frac{OC}{ON} = \frac{1/4 + 1/8}{3/8 + 1/8} = 3/4 =$

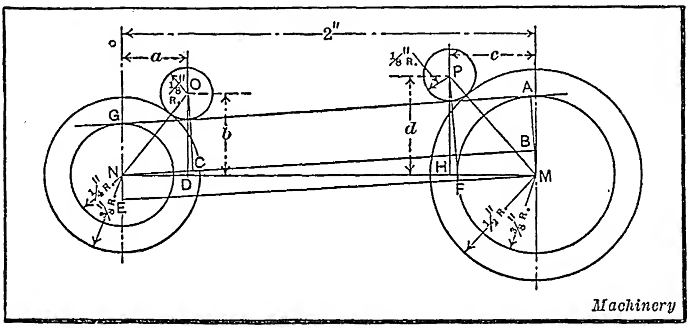

Fig. 17. To find the Center Distances a, b, c and d

0.75; therefore angle NOC = 41 degrees 24 minutes 35 seconds. ONC = 90 degrees − 41 degrees 24 minutes 35 seconds = 48 degrees 35 minutes 25 seconds. $OND = ONC + CND$ = 48 degrees 35 minutes 25 seconds + 3 degrees 35 minutes = 52 degrees 10 minutes 25 seconds. $OD = 1/2 \times \sin OND = 1/2 \times \sin$ 52 degrees 10 minutes 25 seconds $= 1/2 \times 0.78988 = 0.39494$ inch $= b$. $ND = 1/2 \times \cos OND = 1/2 \times \cos$ 52 degrees 10 minutes 25 seconds $= 1/2 \times 0.61327 = 0.30664$ inch $= a$. Draw ME parallel to GA; draw PF and PH perpendicular to ME and MN, respectively. Angle $FPH = EMN = MNB$. $\cos FPM = \frac{1/8 + 1/4 + 1/8}{1/2 + 1/8} = 4/5 = 0.8$; therefore angle FPM = 36 degrees 52 minutes 11 seconds. Angle $HPM = FPM + FPH$ = 36 degrees 52 minutes 11 seconds + 3 degrees 35 minutes = 40 degrees 27 minutes 11 seconds. $HM =$

$\frac{5}{8} \times \sin HPM = \frac{5}{8} \times \sin 40$ degrees 27 minutes 11 seconds $= \frac{5}{8} \times 0.64882 = 0.40551$ inch $= c$. $HP = \frac{5}{8} \times \cos HPM = \frac{5}{8} \times \cos 40$ degrees 27 minutes 11 seconds $= \frac{5}{8} \times 0.76094 = 0.47559$ inch $= d$.

To Locate Intersection of Tapers. The problem is to deduce the general formula for finding the point of intersection of two tapers with reference to measured diameters on those tapers.

Solution: In the diagram, Fig. 18,

L = the distance between the two measured diameters, D and d;

X = the required distance from one measured diameter to the intersection of tapers;

a = angle of long taper;

a_1 = angle of short taper.

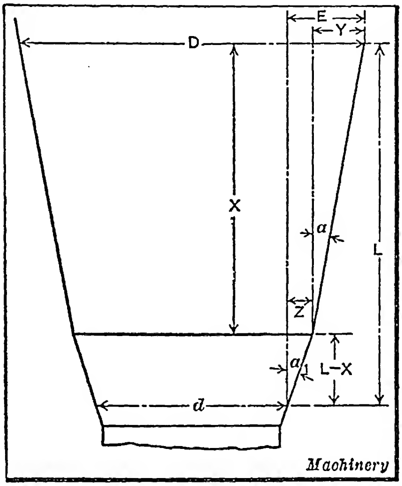

Fig. 18. To find Dimension X from a Given Diameter D to the Intersection of Two Conical Surfaces

Then

$$E = \frac{D - d}{2} = Z + Y$$

$$Z = (L - X) \tan a_1$$
$$Y = X \tan a$$

Therefore:

$$\frac{D - d}{2} = (L - X) \tan a_1 + X \tan a$$

and

$$D - d = 2 \tan a_1 (L - X) + 2X \tan a \qquad (1)$$

But

$$2 \tan a_1 = T_1 \qquad \text{and} \qquad 2 \tan a = T$$

in which T and T_1 represent the long and short tapers per inch, respectively.

Therefore from Equation (1)

$$D - d = T_1 (L - X) + TX$$
$$D - d = T_1 L - T_1 X + TX$$
$$X (T_1 - T) = T_1 L - (D - d)$$

$$X = \frac{T_1 L - (D - d)}{T_1 - T}$$

Formula for Dimension X, Fig. 19. The dimensions D, L and X and angles A and B of the templet, Fig. 19, are to be checked. It

is a comparatively simple matter to check the dimensions D and L and angles A and B, but the checking of the dimension X from the top of the templet to the intersection of the two surfaces which constitute each side, is not so readily accomplished because these

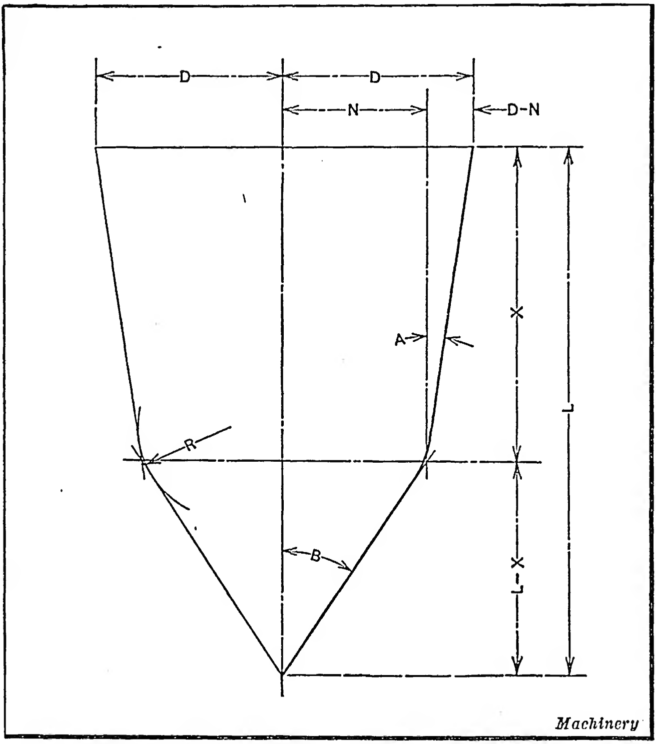

Fig. 19. Diagram used in deducing Formula for finding Dimension X

surfaces are joined together by an arc of which R is the radius. Tolerances are allowed on angles A and B and, of course, dimension X varies as these angles vary.

Solution: The formula for determining dimension X after dimensions D and L and angles A and B are known, is evolved as follows:

In the illustration,

$$\frac{D - N}{X} = \tan A; \qquad \text{hence} \qquad N = D - X \tan A$$

$$\frac{N}{L - X} = \tan B; \qquad \text{hence} \qquad N = \tan B\,(L - X)$$

Therefore

$$D - X \tan A = L \tan B - X \tan B$$

Solving,

$$X\,(\tan B - \tan A) = L \tan B - D$$

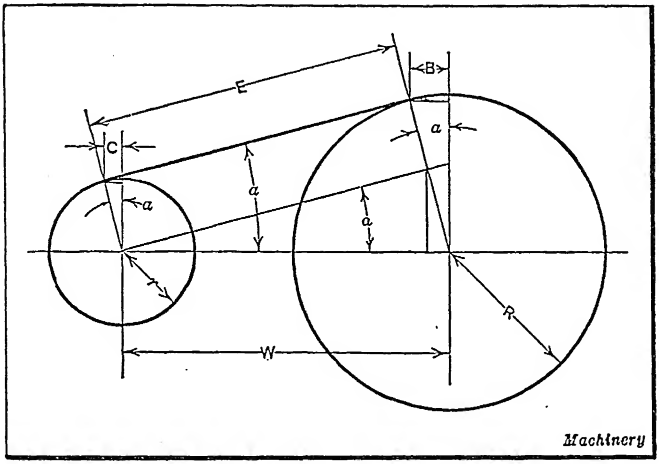

Fig. 20. To find Dimension *E* or Distance between Points of Tangency

$$X = \frac{L \tan B - D}{\tan B - \tan A}$$

This formula can be applied to a large variety of work where similar conditions are encountered.

To Determine Length of Tangent to Two Circles. In tool designing it frequently becomes necessary to determine the length of a tangent to two circles. In Fig. 20,

R = radius of large circle;
r = radius of small circle;
W = center distance between circles; and
a = angle formed between the tangent and the horizontal center line.

With the values given it is required to find the following:

E = length of tangent;

B = length of horizontal line from point of tangency on large circle to the vertical center line; and

C = length of horizontal line from point of tangency on small circle to the vertical center line.

Solution: By trigonometry,

$$\operatorname{Sin} a = \frac{R - r}{W} \qquad \text{and} \qquad E = W \cos a$$

$$B = R \sin a \qquad \text{and} \qquad C = r \sin a$$

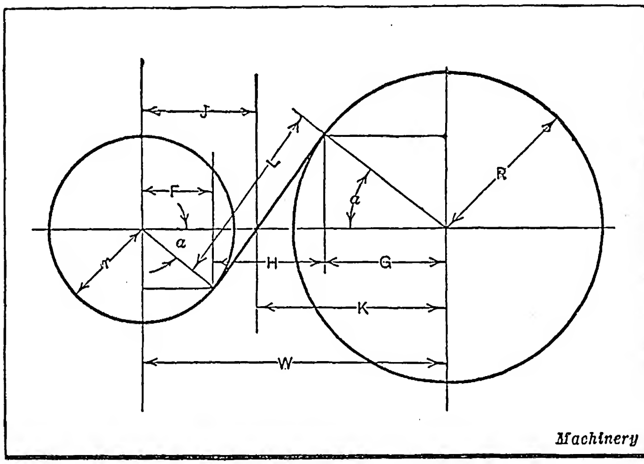

Fig. 21. To find Dimension L

When the tangent crosses a line passing through the centers of both circles, as shown in Fig. 21, the solution is somewhat different. In solving the latter problem let

R = radius of large circle;

r = radius of small circle; and

W = center distance between circles.

The dimensions that are required to be found are:

L = length of tangent;

F = length of horizontal chord from point of tangency on small circle to the vertical center line; and,

G = length of horizontal chord from point of tangency on large circle to the vertical center line.

By trigonometry we have $J = r \sec a$; $K = R \sec a$; and $W = J + K$; or by substitution,

$$W = r \sec a + R \sec a = (r + R) \sec a$$

Now

$$\text{Sec } a = \frac{W}{R + r}.$$

Also

$$F = r \cos a \qquad \text{and} \qquad G = R \cos a$$

By subtraction,

$$H = W - F - G$$

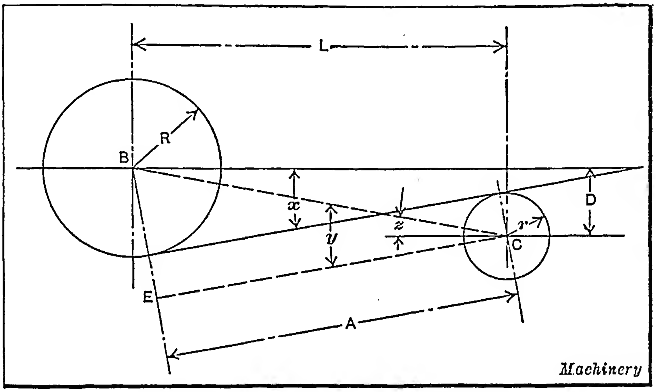

Fig. 22. To find Dimension *A* and Angle *x*

Hence

$$H = W - r \cos a - R \cos a$$

or

$$H = W - (R + r) \cos a$$

Then,

$$L = H \operatorname{cosec} a$$

By substitution,

$$L = [W - (R + r) \cos a] \operatorname{cosec} a$$

These formulas are of considerable value in laying out complicated profile gages.

To Calculate Dimension *A*, Fig. 22, and Angle *x*. This problem is similar to the one illustrated by diagram Fig. 20. It is required to determine the distance between the points of tangency of a line with the studs shown in the illustration, and the angle formed by the line of tangency. If the horizontal and vertical center distances

are known, and also the diameters of the studs, how can the other dimensions be found?

Solution: By the conditions of the problem, L and D are known, also the radii R and r. It is then required to find the value of A, and of angle x.

Draw line BC, also draw line CE parallel to the line of tangency.

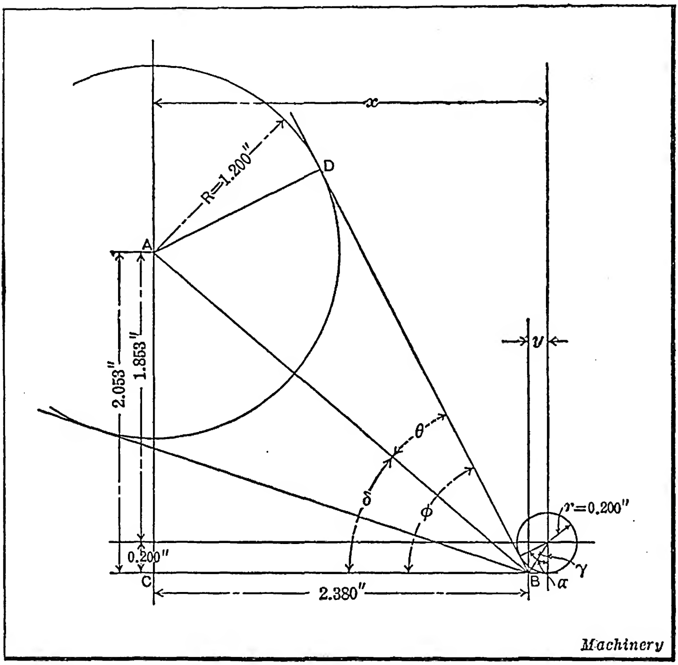

Fig. 23. To find Dimension x, having the Dimensions given on Diagram

$$BC = \sqrt{L^2 + D^2}$$
$$BC^2 = L^2 + D^2$$
$$A = \sqrt{BC^2 - (R + r)^2}$$

Substituting for BC^2 in the last equation:

$$A = \sqrt{(L^2 + D^2) - (R + r)^2}$$

Tan $y = \dfrac{R + r}{A}$ and tan $z = \dfrac{D}{L}$. Knowing the values of y and z, angle $x = y - z$.

To Calculate Dimension x, Fig. 23. In connection with a certain operation in the tool-room it was necessary to determine the distance x between the centers of the two circles shown, the known dimensions corresponding to those given in the illustration.

Solution: In the illustration, angle δ is easily found, for its tangent is:

$$\frac{AC}{BC} = \frac{2.053}{2.380} = 0.8626$$

which is the tangent of an angle of 40 degrees 47 minutes. The side AB of the right triangle ACB is also readily found, being $\sqrt{AC^2 + CB^2}$ $= \sqrt{2.053^2 + 2.380^2}$ $= 3.1431$. Then angle θ may be found, for its sine is:

$$\frac{AD}{AB} = \frac{1.200}{3.1431} = 0.3818$$

which is the sine of an angle of 22 degrees 27 minutes. As angle ϕ is the sum of the two angles just found, ϕ = 40 degrees 47 minutes + 22 degrees 27 minutes = 63 degrees 14 minutes. Angle α = angle ϕ and angle $\gamma = \frac{1}{2}\alpha$; therefore, $\gamma = \frac{1}{2} \times 63$ degrees 14 minutes = 31 degrees 37 minutes. Distance y may now be found, for it is $r \tan \gamma = 0.200 \times 0.61561 = 0.12312$ inch. Therefore, the distance x between the centers of the circles is $0.12312 + 2.38 = 2.5031$ inches.

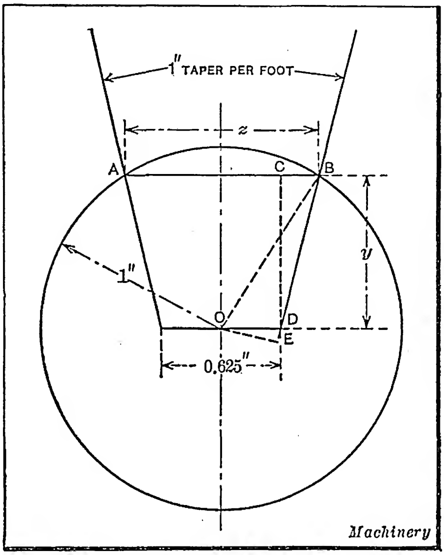

Fig. 24. To find Dimensions y and z

To Find Dimensions y and z, Fig. 24. The end of a part having sides which taper one inch per foot is in contact with a circular object as shown in Fig. 24. What are the values of y and z when the other dimensions correspond to those given in the illustration?

Solution: Extend BD to E, draw DC parallel to the center line, and draw OE at right angles to BE. A taper of 1 inch per foot gives an angle of 2 degrees, 23 minutes, 9 seconds, which is the size of

angles BDC and EOD. Sin $BDC = 0.04163$, which is also sin EOD; and cos $BDC = 0.99913$, which is also cos EOD. As $OD =$ 0.3125 inch, $OE = 0.3125 \times 0.99913 = 0.31223$ inch and $DE =$ $0.3125 \times 0.04163 = 0.013$ inch. $BE = \sqrt{1^2 - 0.31223^2} = 0.95$ in. So $BD = 0.95 - 0.013 = 0.937$ inch. Therefore, CD, or $y =$ $0.937 \times 0.99913 = 0.936$ inch, and AB or $z = 0.625 + (\frac{1}{12} \times 0.93618) = 0.703$ inch.

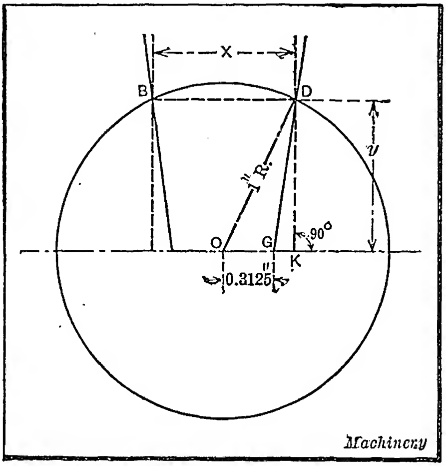

Fig. 25. Diagram for Second Solution of Problem represented by Fig. 24

Another Solution: In Fig. 25, $OG = 0.3125$, the taper of the hole is 1 inch per foot, and the radius of the circle is 1 inch. Draw OD and the altitude DK. As angle $GDK = 2$ degrees, 23 minutes, 9 seconds, angle OGD = 92 degrees, 23 minutes, 9 seconds. According to the law of sines, in any triangle the sides are proportional to the sines of the opposite angles. Therefore, in triangle OGD,

$$\frac{OG}{\sin ODG} = \frac{OD}{\sin OGD};$$

or

$$\sin ODG = \frac{OG \times \sin OGD}{OD} = \frac{0.3125 \times \sin 92 \text{ deg., } 23 \text{ min., } 9 \text{ sec.}}{1}$$

Since sin 92 degrees, 23 minutes, 9 seconds = sin 87 degrees, 36 minutes, 51 seconds (subtracting from 180 degrees) sin ODG = $\frac{0.3125 \times 0.99913}{1} = 0.31222$, from which it is found that angle ODG = 18 degrees, 11 minutes, 35 seconds. Adding angles ODG and GDK gives angle ODK = 20 degrees, 34 minutes, 44 seconds. Then DK, or $y = \cos ODK \times 1$ inch = 0.936 inch and OK or $\frac{1}{2}X$ = sin $ODK \times 1$ inch = 0.35149 inch, making X equal to 0.703 inch.

CHAPTER V

CALCULATING UNKNOWN DIAMETERS OR RADIAL DIMENSIONS

OWING to the extensive use of circular parts or surfaces in connection with practically all forms of mechanical devices, many designing problems, especially in connection with the design of machinery and tools, involve the calculations of diameters or radial dimensions. This chapter contains a diversified collection of such problems which illustrate how the principles of geometry and trigonometry are applied in their solution.

Radius for Given Degree of Curvature. The term "degree of curvature" is confined chiefly to railroad practice. Consequently, many mechanical draftsmen are at a loss as to how to proceed to lay out a curve of a given degree of curvature unless there is available a table of corresponding radii, or a railroad curve with the required degree of curvature and the equivalent radius stamped on it. Briefly, the degree of curvature is the angle at the center of the curve subtended by a chord 100 feet in length. Thus it is evident that the greater the radius the smaller will be the degree of curvature, and vice versa.

The problem of determining the radius for a given degree of curvature is one of simply finding an equation that will give the hypotenuse of a right-angled triangle having an included angle equal to one-half the degree of curvature, and the short side equal to one-half the chord length, or 50 feet. Letting D equal the degree of curvature and R the required radius, we have the formula,

$$R = \frac{50}{\sin (D \div 2)}$$

Thus, for example, if we desire to find the radius R, in feet, of a curve having a degree of curvature of 25 degrees, we have:

$$R = \frac{50}{\sin 12 \text{ degrees } 30 \text{ minutes}} = \frac{50}{0.21644} = 231.01$$

Given the Chord and Arc to Find the Radius. If the length of a given circular arc is $\frac{5}{8}$ inch and the length of the chord is $\frac{9}{16}$ inch, how can the radius be found?

To solve a problem of this kind, first find an approximate value for the central angle $AOB = \phi$ (see Fig. 1); then assume several other angles of about the same value and, finally, calculate ϕ by interpolation. Represent the length of the arc by L and the length of the chord by C; then, for the present case, $C \div L = 9/16 \div 5/8 = 0.9$, and $C = 0.9L$. The table "Segments of Circles" in MACHINERY'S HANDBOOK, gives the length of arc and the length of chord (both to a radius 1) from 1 degree to 180 degrees. Multiplying the value of L for different angles by 0.9 until the product is equal to the value of C for that angle or very nearly equal to it, we find that for $\phi = 90$ degrees, $0.9L = 0.9 \times 1.571 = 1.4139$, and $C = 1.414$ as given in the table. As these two values are practically equal, ϕ is very nearly equal to 90 degrees. The radius r evidently equals

$$\frac{1/2 \times 9/16}{\sin 1/2\phi} = 0.3977.$$

It would not be safe to rely on this value of r to more than three significant figures.

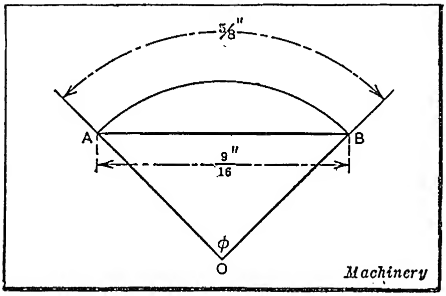

Fig. 1. To find Radius when Chord and Arc are given

Angular Measurements with Disks in Contact. When two disks are in contact with each other, as shown by the diagram, Fig. 2, the diameter D of the larger disk may be determined by the following formula, assuming that angle a and diameter d are known.

$$D = d \times \cot^2 \frac{180 \text{ deg.} - a}{4}$$

in which

D = diameter of the large disk;
d = diameter of the small disk;
a = the angle to be measured.

The following example will illustrate the application of the formula: If the required angle a is 15 degrees and the diameter of the smaller disk is 1 inch, find the diameter D of the larger disk.

Substituting the given values in the formula given, we have:

$$D = 1 \times \cot^2 \frac{180 \text{ deg.} - 15 \text{ deg.}}{4} = 1 \times (1.1403)^2 = 1.3002$$

This formula may be given in the form of a rule as follows: Subtract the required angle from 180 degrees, divide by 4, find the cotangent corresponding to this angle in a table of trigonometrical functions and multiply the square of this cotangent by the diameter of the small disk.

The value of D may also be determined by another formula in which the sine of angle a is used. Thus,

$$D = \frac{2d\left(\sin\frac{a}{2}\right)}{1 - \sin\frac{a}{2}} + d$$

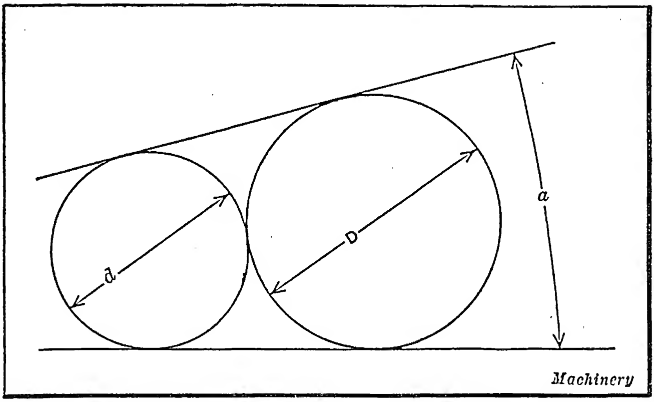

Fig. 2. To find Diameter D, when Diameter d and Angle a are given

From this we get

$$D = \frac{2d\left(\sin\frac{a}{2}\right) + d - d\left(\sin\frac{a}{2}\right)}{1 - \sin\frac{a}{2}} = \frac{d\left(\sin\frac{a}{2} + 1\right)}{1 - \sin\frac{a}{2}}$$

or, expressed in conventional form

$$D = d \times \frac{1 + \sin\frac{a}{2}}{1 - \sin\frac{a}{2}}$$

If d equals 1 inch and a equals 15 degrees,

$$D = 1 \times \frac{1.13053}{0.86947} = 1.3002 \text{ inches}$$

It may be of interest to point out that the diameter of the tangent circle P (see Fig. 3) on the opposite side of circle d, is the reciprocal of the diameter D, that is

$$P = d \times \frac{1 - \sin\frac{a}{2}}{1 + \sin\frac{a}{2}} = 1 \times \frac{0.86947}{1.13053} = 0.76908 \text{ inch}$$

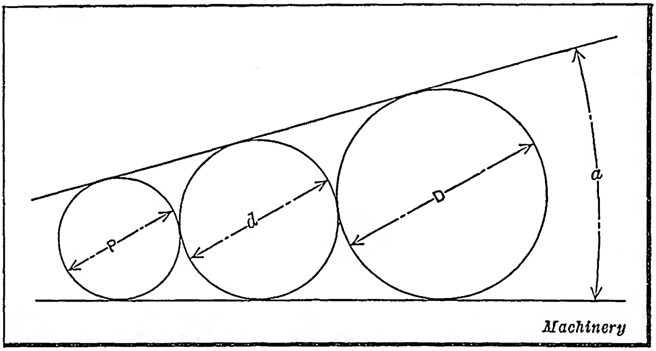

Fig. 3. To find Diameter P of Tangent Circle

From the foregoing it follows that the product of the diameters of the two tangent circles P and D is constant and is equal to the square of the diameter of circle d.

To Calculate the Outer Radius of a Hollow Sphere. If the shell of a hollow iron ball is 4 inches thick, and contains one-fifth the cubical contents of the whole ball, how can the outer radius of the ball be found?

Solution: Let R be the outer radius of the ball; then $R - 4$ will be the radius of the hollow sphere enclosed by the shell. As the volume of a sphere is proportional to the cube of its radius, the conditions of the problem are given by the equation:

$$R^3 - (R - 4)^3 = \tfrac{1}{5} R^3$$

or

$$\tfrac{4}{5} R^3 = (R - 4)^3$$

Extracting the cube root, transposing, and combining,

$$R = \frac{4}{1 - \sqrt[3]{\frac{4}{5}}}$$

Since $\frac{4}{5} = \frac{8}{10}$ and 8 is a perfect cube, the above equation may be written

$$R = \frac{4\sqrt[3]{10}}{\sqrt[3]{10} - 2}$$

By reference to a table of cube roots the proper values can be easily ascertained, and then, by working out the formula, it will be found that $R = 55.80$ inches.

Radius of Circle Tangent to Side AB, Fig. 4, and Intersecting Points C and E. In the illustration, ABC is a right triangle, right-angled at B. With the dimensions given, it is required to find the radius of a circle that will pass through points C and E and be tangent to the side AB.

Solution: Following is a simple method of finding the radius of this circle.

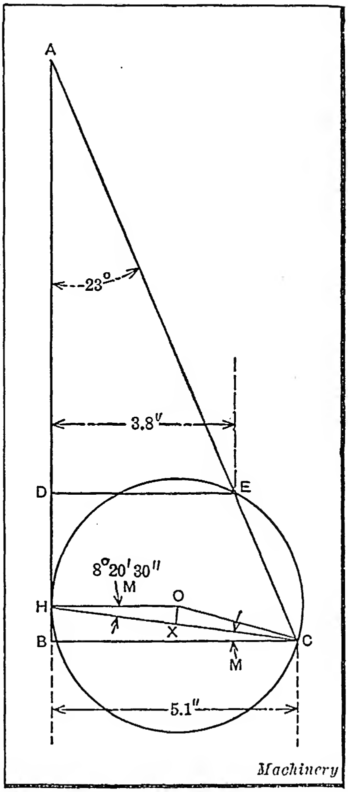

Fig. 4. To find Radius of Circle intersecting Points C and E

$$AE = \frac{DE}{\sin 23 \text{ deg.}}$$

$$= \frac{3.8}{0.39073} = 9.7254$$

and

$$AC = \frac{BC}{\sin 23 \text{ deg.}}$$

$$= \frac{5.1}{0.39073} = 13.0525$$

By geometry, $AC \times AE = (AH)^2$, so

$$AH = \sqrt{9.7254 \times 13.0525} = 11.267$$

$$AB = BC \times \cot 23 \text{ degrees} = 5.1 \times 2.3558 = 12.01458$$

$$HB = AB - AH = 0.74758$$

$$\text{Tan } M = \frac{HB}{BC} = \frac{0.74758}{5.1} = 0.14658$$

$$M = 8 \text{ degrees, } 20 \text{ minutes, } 20 \text{ seconds}$$

$$HC = \frac{BC}{\cos 8 \text{ deg., } 20 \text{ min., } 20 \text{ sec.}} = \frac{5.1}{0.98943} = 5.1545$$

$$HO = \frac{0.5HC}{\cos 8 \text{ deg., } 20 \text{ min., } 20 \text{ sec.}} = \frac{2.5773}{0.98943} = 2.6048 \text{ inches}$$

which is the radius desired.

To Find Radius of Circle Inscribed in Triangle. In a right triangle having the dimensions shown in Fig. 5, a circle is inscribed. Find the radius of the circle.

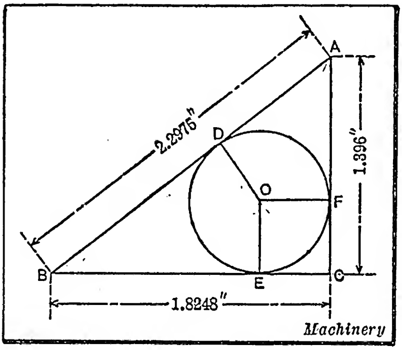

Fig. 5. To find Radius of Circle inscribed in Triangle

Solution: In the illustration, $BD = BE$ and $AD = AF$, because "Tangents drawn to a circle from the same point are equal." $EC = CF$, and EC = radius OF. Then let R = radius of inscribed circle. $AC - R = AD$ and $BC - R = DB$. Adding,

$$AC + BC - 2R = AD + DB$$

$$AD + DB = AB$$

hence,
$$AC + BC - AB = 2R$$

Stated as a rule, "The diameter of a circle inscribed in a right triangle is equal to the difference between the hypotenuse and the sum of the other sides." Substituting the given dimensions, we have $1.396 + 1.8248 - 2.2975 = 0.9233 = 2R$, and $R = 0.4616$.

Radius of Disk in Contact at Three Points as in Fig. 6. The diagram shows a disk which is laid against two surfaces located at right angles to each other. If a size block of known dimensions is placed as shown, how can the radius R of the disk be calculated, the disk being tangent to both sides, and to the corner of the block?

Solution: According to the diagram:

$$R^2 = (R - A)^2 + (R - B)^2$$

Expanding the above equation and solving for R,

$$R = A + B \pm \sqrt{2AB}$$

If we assume the dimensions of the size block to be A = ¼ inch and B = ½ inch, and then apply the formula given,

$$R = 0.25 + 0.5 + \sqrt{2 \times 0.25 \times 0.5}$$

$$= 0.75 + \sqrt{0.25} = 1.25 \text{ inches}$$

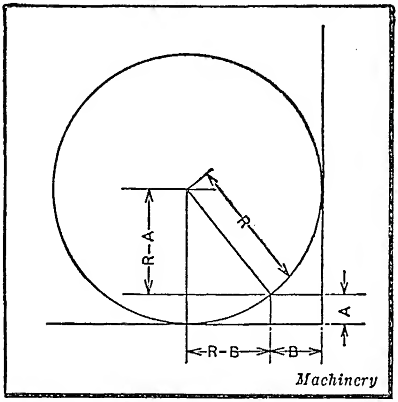

Fig. 6. To find Radius of Circle in Contact at Three Points as shown

It is obvious that the conditions of the problem do not warrant using the minus value in the above algebraic equation as this condition does not apply.

Second Solution: This problem can be solved readily and simply by means of analytical geometry as follows: Draw the coordinates x and y, Fig. 7, from any point on the circumference of the circle, as M. Take the general equation of the circle

$$(x - a)^2 + (c - y)^2 = r^2$$

in which a and c are the coordinates of the center of a circle of radius r. According to the conditions of the problem $a = c = r$, and by substitution in the equation of the circle

$$(x - r)^2 + (r - y)^2 = r^2$$

or $r^2 - 2(x + y)r + x^2 + y^2 = 0$

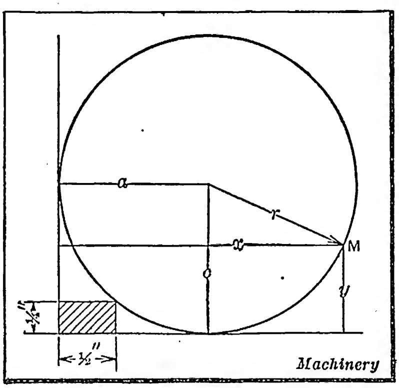

Fig. 7. Diagram for Second Solution of Problem represented by Fig. 6

Assume that the dimensions of the block are the same as before, that is, ¼ inch thick and ½ inch wide; then, substituting the values just given in place of the coordinates x and y, and solving this

quadratic equation, the value of r can be readily determined. Thus,

$$r^2 - 1\tfrac{1}{2}r + \tfrac{5}{16} = 0 \qquad \text{and} \qquad r = \tfrac{3}{4} \pm \sqrt{\tfrac{9}{16} - \tfrac{5}{16}}$$

Hence, $r = \frac{1}{4}$ inch or $1\frac{1}{4}$ inches.

It is quite obvious that of the two roots of the quadratic equation, the larger one applies to the conditions given.

Third Solution: By geometry, if from a point without a circle, a secant and a tangent are drawn, as AC and AD (Fig. 8), the tangent is the mean proportional between the whole secant and its exterior section. Hence

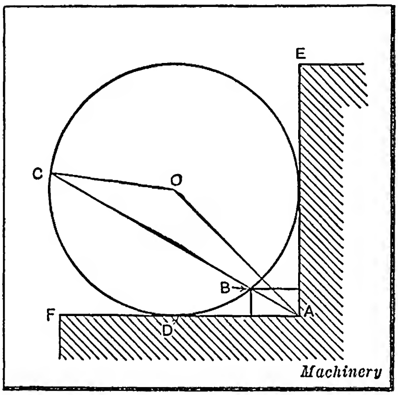

Fig. 8. Diagram for Third Solution of Problem represented by Fig. 6

$$AC = \frac{AD^2}{AB}$$

But $AD = R$ = radius, and by trigonometry AC

$$= \frac{R \sin COA}{\sin OAC}\text{; therefore:}$$

$$\frac{R^2}{AB} = \frac{R \sin COA}{\sin OAC}$$

Simplifying,

$$R = \frac{AB \sin COA}{\sin OAC}$$

Angle OAD = 45 deg.

$$\text{Tan angle } DAB = \frac{0.25}{0.50}$$

$$DAB = 26 \text{ deg. } 33 \text{ min. } 55 \text{ sec.}$$

Angle OAC = 45 deg. − 26 deg. 33 min. 55 sec. = 18 deg. 26 min. 5 sec.

$$OA = 1.414R$$

then by the law of sines,

$$\frac{1.414R}{R} = \frac{\sin OCA}{\sin OAC}$$

$$\sin OCA = 1.414 \sin OAC = 1.414 \times 0.31622 = 0.44714$$

$$OCA = 26 \text{ degrees } 33 \text{ minutes } 37 \text{ seconds}$$

$$COA = 180 \text{ deg.} - (OAC + OCA) = 180 \text{ deg.} - (18 \text{ deg. } 26 \text{ min. } 5 \text{ sec.} + 26 \text{ deg. } 33 \text{ min. } 37 \text{ sec.}) = 135 \text{ deg. } 18 \text{ sec.}$$

$$AB = \sqrt{0.25^2 + 0.50^2} = 0.559 \text{ inch}$$

Therefore

$$R = \frac{0.559 \times 0.70705}{0.31622} = 1.250 \text{ inch}$$

and the diameter of the disk equals 2 × 1.25, or 2.50 inches.

How to Find Radius r of Disk, Fig. 9. The problem is to find the radius of a circular disk which is in contact with horizontal and vertical surfaces when the disk just touches the corner A of a gage-block. The two dimensions given are on the diagram.

Solution: This problem may be solved by means of plane trigonometry. From the diagram it will be seen that AB = ¼ inch, and $BC = 1$ inch. Therefore:

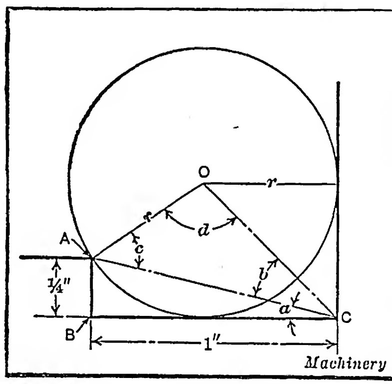

Fig. 9. To find Radius r

$$\frac{AB}{BC} = 0.250 = \tan a$$

a = 14 degrees 2 minutes 10 seconds

and

$$b = 45 \text{ degrees} - a = 30 \text{ deg. } 57 \text{ min. } 50 \text{ sec.}$$

$$AC = \sqrt{(AB)^2 + (BC)^2} = \sqrt{1.0625} = 1.0308 \text{ inches}$$

or

$$AC = \sec a \times 1 \text{ inch} = 1.0308 \text{ inches}$$

From a corollary in mensuration (see Machinery's Handbook) $OC = 1.414r$. According to the law of sines:

$$\frac{r}{1.414r} = \frac{\sin b}{\sin c}$$

from which

$$\sin c = 1.414 \times \sin b = 0.7275$$

and

$$c = 46 \text{ degrees } 40 \text{ minutes } 39 \text{ seconds}$$

Therefore

$$d = 180 \text{ deg.} - (b + c) = 102 \text{ deg. } 21 \text{ min. } 31 \text{ sec.}$$

Then

$$\frac{r}{AC} = \frac{\sin b}{\sin d}$$

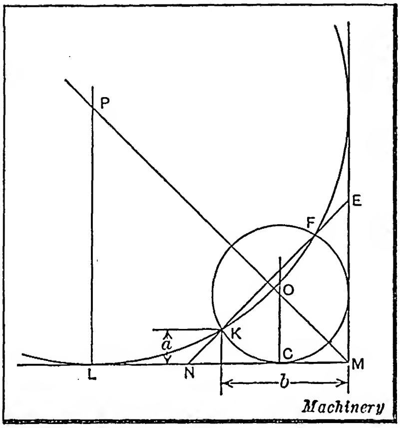

Fig. 10. Diagram for Second Solution of Problem represented by Fig. 9

from which

$$r = \frac{AC \sin b}{\sin d}$$

$$= \frac{1.0308 \times 0.5145}{0.97683}$$

$$= 0.5429 \text{ inch}$$

and the diameter of the circle, $2 \times 0.5429 = 1.0858$ inches.

Second Solution: Another method of solving this problem will be explained in connection with Fig. 10. Through point K draw NE at 45 degrees from surface NM. $KE = b\sqrt{2}$, and $NK = a\sqrt{2}$. It follows then that, since $KE = NF$,

$$NC = \sqrt{b\sqrt{2} \times a\sqrt{2}} = \sqrt{2ab}$$

If, as in the previous examples, $a = \frac{1}{4}$ inch and $b = 1$ inch, then

$$NC = \sqrt{0.5}$$
$$= 0.707106 \text{ inch}$$
$$CM = 1.25 - 0.707106$$
$$= 0.5429 \text{ inch}$$

It is evident that many mathematical problems can be solved by the application of a number of different methods, and that, therefore, the same results may be obtained in several different ways. It is not always possible to state that any one method is the best; to some, one method may seem easier, to others, another.

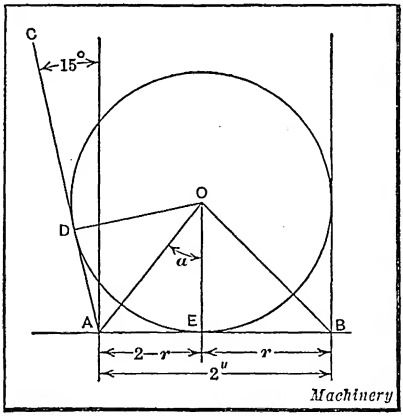

Fig. 11. To find Diameter of Circle Tangent to Three Sides as shown

Considering again the condition represented by Figs. 6 to 8 inclusive, in which the disk touches the required surfaces and also passes through the point K as shown by the larger arc in the diagram, Fig. 10, the radius may be found by laying off NL equal to NC and erecting perpendiculars at both points L and C as shown. Then by passing a 45-degree diagonal from point M through these perpendiculars, the intersection with each will be the center of the required disks. The radius, then, of the large disk may be found simply by multiplying NC by 2 and adding CM.

Diameter of Plug which is Tangent to Three Sides, Fig. 11. Find the correct diameter of a plug which will be tangent to the three sides of a gage as shown in the illustration. The 2-inch dimension and the 15-degree angle are all that are known.

Solution: Angle CAB = 90 degrees + 15 degrees = 105 degrees. Angle DOE + angle CAB = 180 degrees. Therefore angle DOE = 180 − 105 = 75 degrees. Since line OA bisects angle DOE, angle $a = \frac{1}{2} \times 75 = 37$ degrees 30 minutes.

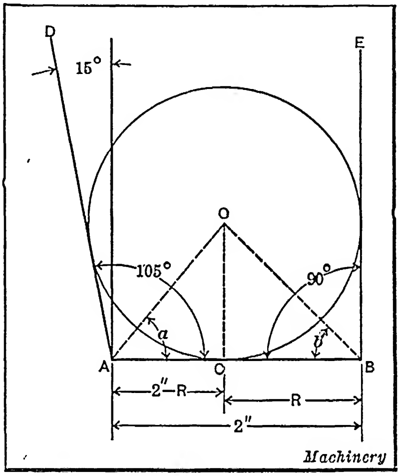

Fig. 12. Diagram for Second Solution of Problem represented by Fig. 11

$$\frac{2-r}{r} = \tan 37 \text{ degrees } 30 \text{ minutes}$$

$$r(\tan a + 1) = 2 \qquad r = \frac{2}{\tan a + 1}$$

$$\tan a + 1 = 1.76733$$

Therefore,

$$\frac{2}{1.76733} = r = 1.13165 \text{ inch} \qquad \text{and} \qquad 2r = 2.2633 \text{ inches}$$

Another Solution: Following is another simple solution of this problem. Draw OA and OB (Fig. 12) which, according to geometry, bisect the angles DAB and EBA, respectively. Draw OC perpen-

dicular to AB. In the triangles OAC and OBC, side OC is common. Therefore:

$$\tan a \times (2 - R) = \tan b \times R$$

$$\tan 52 \text{ degrees } 30 \text{ minutes } (2 - R) = \tan 45 \text{ degrees} \times R$$

$$1.30323\ (2 - R) = R$$

$$2.60646 = R + 1.30323R = R\ (1 + 1.30323)$$

$$R = \frac{2.60646}{2.30323} = 1.13165 \text{ inch} \qquad \text{and} \qquad 2R = 2.2633 \text{ inches}$$

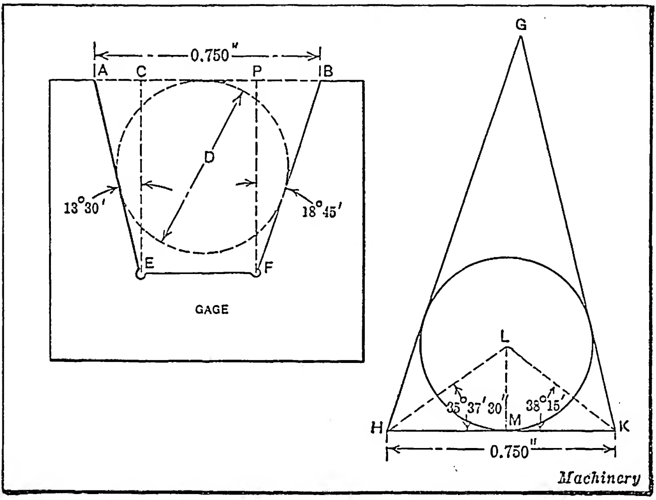

Fig. 13. To find Diameter D

To Determine Diameter D of Plug, Fig. 13. The problem is to determine the diameter D of a plug, which when laid in the gage, will be exactly flush with its top.

Solution: First construct the diagram shown at the right, making line HK equal to the distance from A to B, or 0.750 inch; then from point H draw line HG, making angle GHK equal to angle FBP. The magnitude of this angle is found by adding 90 degrees to 18 degrees 45 minutes, and subtracting the sum from 180 degrees, which gives 71 degrees 15 minutes. In a similar manner, determine angle EAC, which is found to be 76 degrees 30 minutes. Now draw line KG, making angle HKG equal to 76 degrees 30 minutes. Next draw two lines bisecting angles GHK and GKH, respectively, and

through L, the point of intersection of the two bisecting lines, draw LM perpendicular to HK. It is evident that LM will be the radius of an inscribed circle and therefore will equal one-half the required diameter D. In triangle HLK, we have three known elements: Angle LHK equals 35 degrees 37 minutes 30 seconds, or one-half GHK; angle LKH equals 38 degrees 15 minutes; and angle HLK equals 106 degrees 7 minutes 30 seconds (180 degrees − 35 degrees 37 minutes 30 seconds − 38 degrees 15 minutes). Now in the oblique triangle HLK,

$$HL = \frac{0.750 \times \sin 38 \text{ deg. } 15 \text{ min.}}{\sin 106 \text{ deg. } 7 \text{ min. } 30 \text{ sec.}}$$

Then in triangle HLM, side LM equals side $HL \times \sin$ 35 degrees 37 minutes 30 seconds. From these two equations it is evident that

$$LM = \frac{0.750 \sin 38 \text{ deg. } 15 \text{ min.}}{\sin 106 \text{ deg. } 7 \text{ min. } 30 \text{ sec.}} \times \sin 35 \text{ deg. } 37 \text{ min. } 30 \text{ sec.}$$

and as LM equals one-half the required diameter,

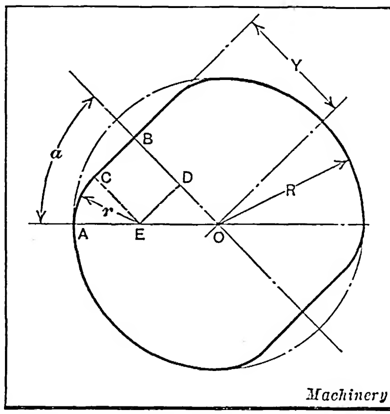

Fig. 14. To find Radius r having Radius R, Angle a and Dimension Y

$$D = 2 \times \frac{0.750 \sin 38 \text{ deg. } 15 \text{ min.} \times \sin 35 \text{ deg. } 37 \text{ min. } 30 \text{ sec.}}{\sin 106 \text{ deg. } 7 \text{ min. } 30 \text{ sec.}}$$

from which

$$D = \frac{1.5 \times 0.61909 \times 0.58248}{0.96066} = 0.563062 \text{ inch}$$

Solution with Logarithms. To solve the same problem using six-place logarithms, proceed as follows:

Log 1.5	0.176091
Log sin 38 deg. 15 min.	9.791757 − 10
Log sin 35 deg. 37 min. 30 sec.	9.765279 − 10
Colog sin 106 deg. 7 min. 30 sec.	0.017431
Log diameter	19.750558 − 20
Diameter	0.563064 inch

From this example it will be clearly seen that the logarithmic method is simpler and easier than the arithmetical method.

To Find the Radius of a Tangent Arc. How can the radius r of a tangent arc for a slabbed cylindrical piece be found if the radius R (Fig. 14) of the piece, the angle a at which the slab is taken, and the distance Y are known?

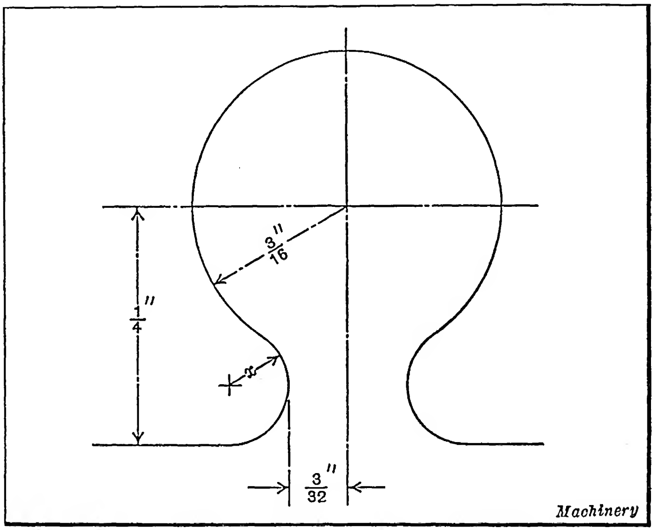

Fig. 15. To find Radius x

Solution: In the diagram draw CE parallel to BO; also draw ED parallel to CB.

Then

$$r = CE = AE = BD$$
$$OD = Y - r \qquad OE = R - r$$

Hence

$$\frac{Y - r}{R - r} = \cos a$$
$$(R - r)\cos a = Y - r$$
$$r - r\cos a = Y - R\cos a$$
$$r = \frac{Y - R\cos a}{1 - \cos a}$$

To Calculate Radius x, Fig. 15. The illustration shows a problem in connection with the designing of dies for blanking a sheet-metal piece, the problem being to find radius x from the other dimensions given.

Solution: From the illustration, we have

$$(x + 3/16)^2 = (1/4 - x)^2 + (x + 3/32)^2$$

Then,

$$\left(\frac{16x + 3}{16}\right)^2 = \left(\frac{1 - 4x}{4}\right)^2 + \left(\frac{32x + 3}{32}\right)^2$$

Expanding, clearing of fractions, and combining,

$$1024x^2 - 704x + 37 = 0$$

This may be written

$$1024x^2 - 2\,(352)\,x + 37 = 0$$

Solving this quadratic equation,

$$x = \frac{352 \pm \sqrt{352^2 - 1024 \times 37}}{1024} = \frac{352 \pm 64\sqrt{21}}{1024}$$

$$x = \frac{11 \pm 2\sqrt{21}}{32}$$

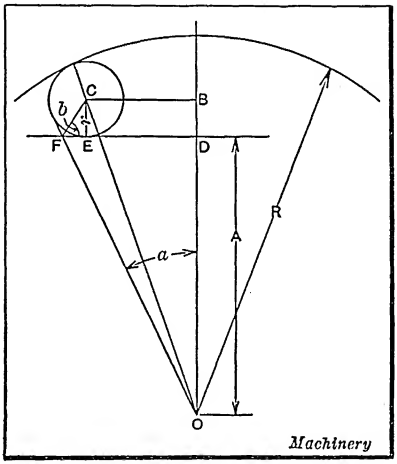

Fig. 16. To find Radius r having Radius R, Dimension A and Angle a

Taking the minus sign before the radical, in this case, we find

$$x = \frac{11}{32} - \frac{\sqrt{21}}{16} = 0.34375 - 0.28641 = 0.05734 \text{ inch.}$$

To Find Radius r, Fig. 16, when Radius R, Dimension A, and Angle a, are Given. A problem encountered in the design of a field pole tip is illustrated by Fig. 16. The only known dimensions are the radius R, distance A, and the size of the angle a. It is required to find radius r of the arc.

Solution: This problem may be solved in the following manner: In the diagram, let A = 7 inches; R = 10 inches; and angle a = 33 degrees 38 minutes 40 seconds.

$$CB^2 = OC^2 - OB^2 = (R - r)^2 - (A + r)^2 = (10 - r)^2 - (7 + r)^2$$

Therefore $$CB = \sqrt{51 - 34r}$$

In the triangle CFE, $FE = r \cot b$
but

$$b = \frac{90 \text{ deg.} + a}{2} = 61 \text{ degrees } 49 \text{ minutes } 20 \text{ seconds}$$

Therefore $$FE = 0.5357r$$

In triangle FDO

$$FD = A \times \tan a = 7 \times 0.66552 = 4.6586 \text{ inches}$$

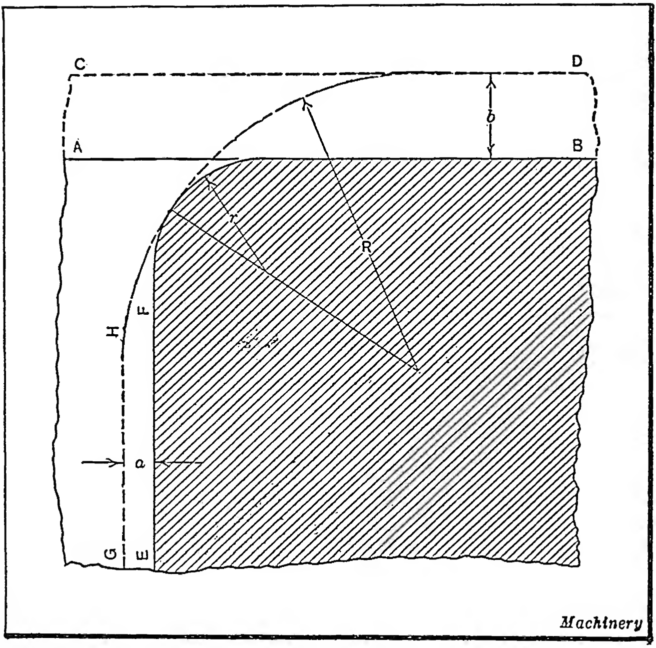

Fig. 17. To find Radius R having Radius r and Dimensions a and b

But $FD = FE + CB$; therefore by substituting the values previously found in this equation and solving for r,

$$4.6586 = 0.5357r + \sqrt{51 - 34r}$$

Transposing and squaring both sides of the equation

$$21.70255 - 4.99122r + 0.28697r^2 = 51 - 34r$$

from which $r = 1$, almost exactly.

To Find Radius R, Fig. 17, Having Radius r and Dimensions a and b. In making tapered roller bearing cones, it sometimes happens that the cone is required to be ground accurately to a given radius r, Fig. 17. AB represents the back face of the finished cone, CD the back face of the cone before grinding, EF the finished bore, and GH the bore before finishing. In cases of this kind it is obviously important that the proper allowance be made for finish-grinding. The destructive effect of a sharp point or edge of steel on the face of the grinding wheel makes it desirable to so proportion the cone that as little stock as possible will need to be removed at the point where the back face joins radius r. The problem of determining radius R occurs when it is planned to make the radius of the unfinished forging and the finished radius tangent to each other in order to eliminate as far as possible the use of the wheel where the destructive effect is greatest. The problem, therefore, consists in determining the radius R, of a circle which is tangent to a given smaller circle, and to two lines at right angles to each other at given unequal distances from tangents to the smaller circle, as indicated in Fig. 18. In the illustration, the dimensions a, b, and r are known.

Solution: In solving this problem, we may start out with three equations, assuming three unknown quantities R, y, and z, Fig. 18. From these three equations we obtain by substitution the final equation in which R is the only unknown quantity. Note that $x = \sqrt{y^2 + z^2}$.

We then have the three equations:

$$R = x + r = \sqrt{y^2 + z^2} + r \tag{1}$$

$$R = y + r + a \tag{2}$$

$$R = z + r + b \tag{3}$$

Transposing,

$$z = R - r - b$$

Then, from Equation (1),

$$R = \sqrt{y^2 + (R - r - b)^2} + r$$

Transposing terms in Equation (2) we have,

$$y = R - r - a$$

Therefore

$$R = \sqrt{(R - r - a)^2 + (R - r - b)^2} + r$$

It is possible to solve this equation for R by transposing r to the left-hand side of the equation and then squaring both sides, which

will result in a quadratic equation. In a problem of this type, however, the trial method will be equally rapid — that is, a trial value may be substituted for R that is as nearly exact as it is possible to determine by measuring a scale drawing. The substituted value is then slightly adjusted until it satisfies the equation.

Having Dimensions Given in Fig. 19 to Find Radius r. In the accompanying diagram, Fig. 19, are shown the conditions encoun-

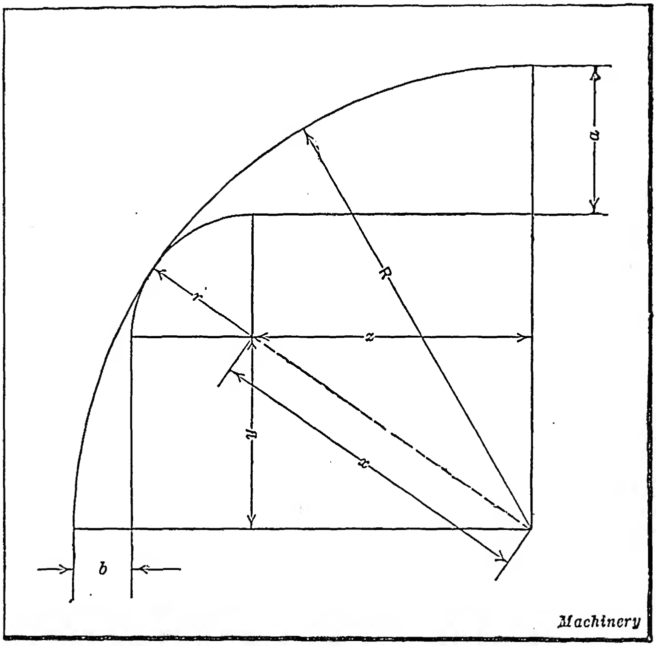

Fig. 18. Diagram used in Solution of Tangency Problem represented by Fig. 17

tered in the design of a roller clutch. From the dimensions given, the problem is to find the radius of the roll.

Solution: From the diagram, it will be evident that

$$OK = \sqrt{1.25^2 + 0.25^2} = 1.2747$$

$$\text{Sin } OKA = \frac{0.25}{1.2747} = 0.19612$$

Hence $$OKA = 11 \text{ degrees } 19 \text{ minutes}$$

By construction, angle $AKH = 60$ degrees; therefore

$$OKH = 60 \text{ deg.} + 11 \text{ deg. } 19 \text{ min.} = 71 \text{ deg. } 19 \text{ min.}$$

and $$HK = r \times \sec 30 \text{ degrees} = r \times 1.1547$$

Also $$OH = 2 - r$$

According to the equation given in MACHINERY'S HANDBOOK

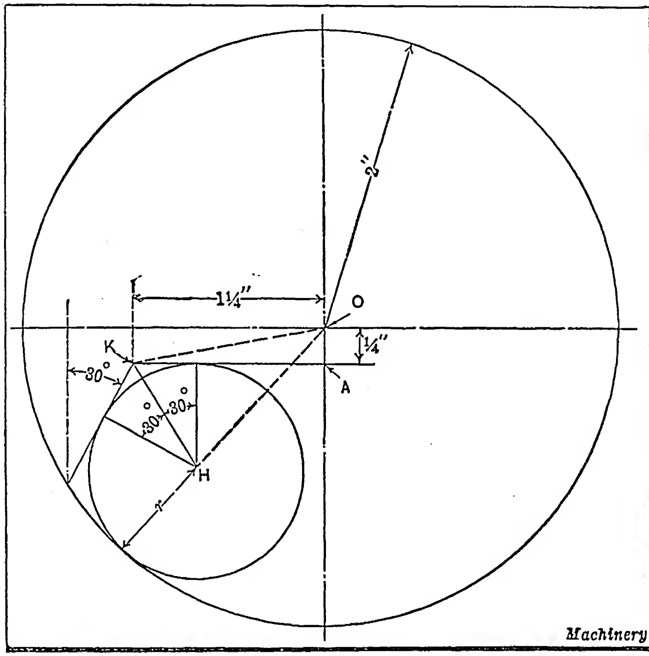

Fig. 19. To find Radius r, having Dimensions given on Diagram

for solving an obtuse-angled triangle, when two sides and the angle between them are known, we have:

$$OH \text{ or } 2 - r = \sqrt{HK^2 + OK^2 - 2\,(HK \times OK \times \cos OKH)}$$

Substituting the numerical values of the known quantities,

$$2 - r = \sqrt{(1.1547r)^2 + 1.2747^2 - 2\,(1.1547r \times 1.2747 \times 0.32034)}$$

$$2 - r = \sqrt{1.3333r^2 + 1.6248 - 2\,(1.1547r \times 0.40833)}$$

$$= \sqrt{1.3333r^2 - 0.9430r + 1.6248}$$

Squaring the left-hand member, we have,

$$4 - 4r + r^2 = 1.333r^2 - 0.943r + 1.625$$

Transposing and uniting terms

$$0.333r^2 + 3.057r = 2.375$$

or

$$r^2 + 9.18r = 7.1321$$

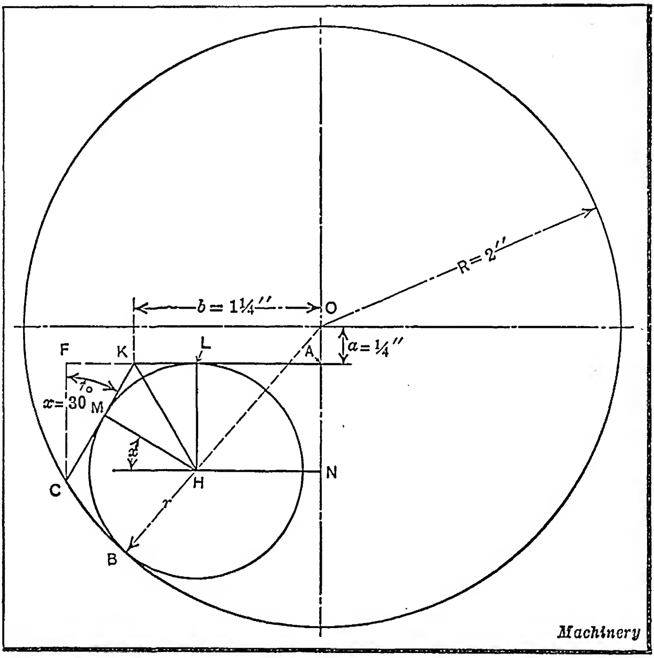

Fig. 20. Diagram for Second Solution of Problem represented by Fig. 19

Solving this quadratic equation according to the formula given in MACHINERY'S HANDBOOK,

$$r = -\frac{9.18}{2} \pm \sqrt{\frac{9.18^2}{4} + 7.1321}$$

$$r = -4.59 \pm 5.3104$$

$$r = 0.7204 \text{ inch}$$

Hence the diameter of the roll is equal to $2r$ or 1.4408 inches.

Second Solution: From the diagram Fig. 20 the radius of the roll can also be found in the following manner: Since, by construction, $MK = KL$, angle KHM = angle $KHL = \frac{1}{2}$ (90 degrees $- x$)

Let $\frac{1}{2}$ (90 degrees $- x) = y$; then,

$$KL = r \tan y$$

and

$$LA = HN = b - KL; \qquad HN = b - r \tan y$$

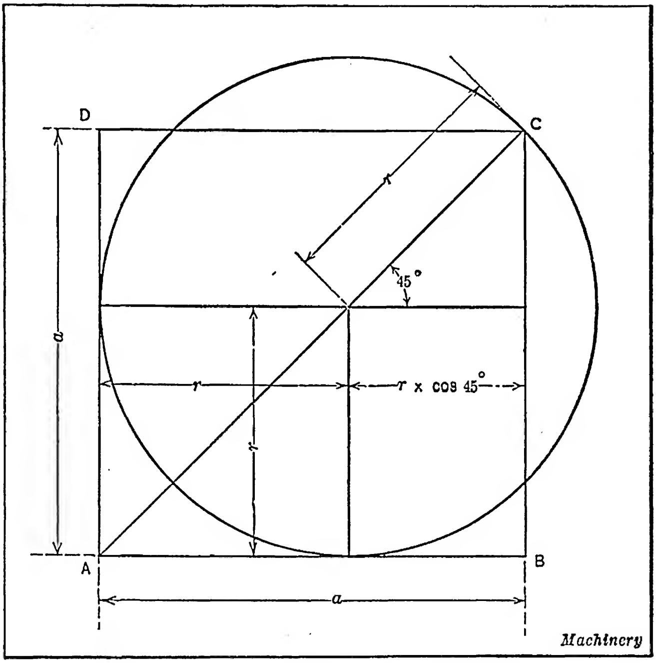

Fig. 21. To find Radius r, having Dimension a

$$HO = \sqrt{(HN)^2 + (NO)^2}$$

But

$$HO = R - r$$

Therefore

$$\sqrt{(b - r \tan y)^2 + (a + r)^2} = R - r$$

Squaring both sides of the preceding equation,

$$(b - r \tan y)^2 + (a + r)^2 = (R - r)^2$$

Expanding, combining, and arranging terms with reference to r, we get the quadratic equation

$$r^2 \tan^2 y + 2r [(R + a) - b \tan y] - (R^2 - a^2 - b^2) = 0$$

Solving this equation for r,

$$r = \frac{-[(R + a) - b \tan y]}{\tan^2 y} \pm \frac{\sqrt{(R + a)^2 - (R + a) \tan y [2b - (R - a) \tan y]}}{\tan^2 y}$$

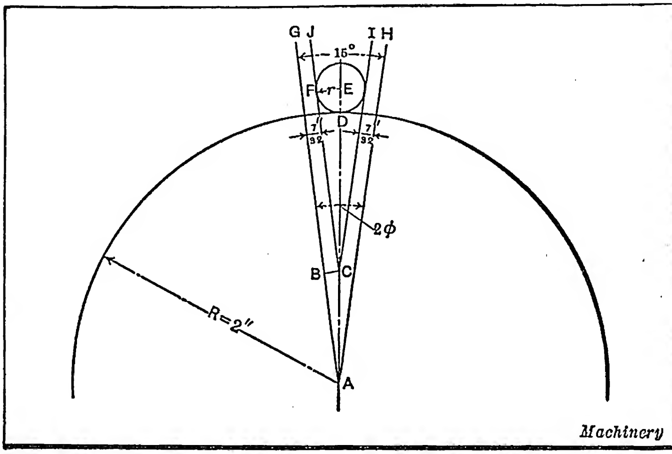

Fig. 22. To find Radius r, having the Values given on Diagram

Now, $y = \frac{1}{2}$ (90 deg. $- x$) = 30 degrees. Tan 30 deg. $= \frac{1}{3}\sqrt{3}$ and $\tan^2$ 30 deg. $= \frac{1}{3}$. Inserting these values of tan y and $\tan^2 y$ and those given in the diagram, in the foregoing equation, and taking the plus value of the quantity under the radical sign, we find that

$$r = \frac{\sqrt{3.12740} - 1.52831}{0.33333}$$

$$= 3(1.76844 - 1.52831)$$

$$= 3 \times 0.24013 = 0.7204 \text{ inch}$$

To Find Radius r, Fig. 21, having Dimension a. In the diagram, the problem is to calculate the radius r of a circle which passes through point C of square $ABCD$ and is tangent to the sides AD and AB. The dimension a of the square is known.

Solution: From the diagram,

$$a = r + r \times \cos 45 \text{ degrees}$$
$$= r\,(1 + \cos 45 \text{ degrees})$$

Hence

$$r = \frac{a}{1 + \cos 45 \text{ degrees}}$$

To Find Radius r, Fig. 22. Find the radius r of small circle, from the dimensions given in the illustration and deduce a formula for all such cases.

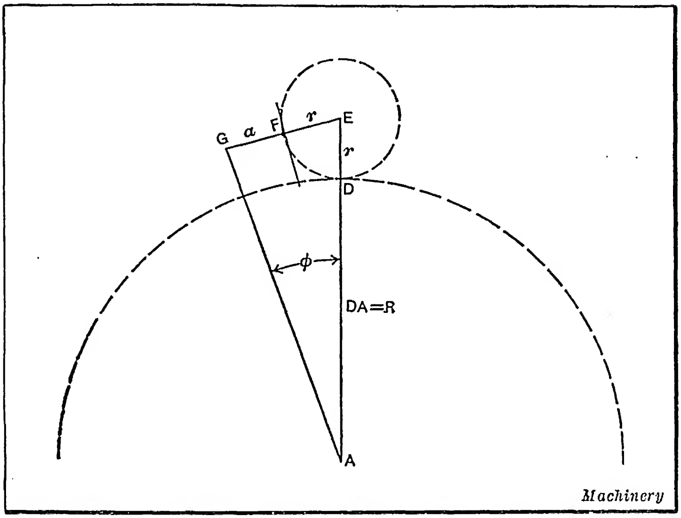

Fig. 23. Diagram for Second Solution of Problem represented by Fig. 22

Solution: Draw CB perpendicular to AB and EF perpendicular to CF. Let R = radius AD, r = radius EF, $a = BC = 0.21875$ inch, and 2ϕ = angle $GAH = JCI$; then, $\phi = BAC = FCE = \frac{15 \text{ deg.}}{2} = 7$ degrees 30 minutes.

$$CA = \frac{a}{\sin \phi} \qquad CD = R - \frac{a}{\sin \phi} = \frac{R \sin \phi - a}{\sin \phi}$$

In the right triangle EFC,

$$\frac{r}{EC} = \frac{r}{r + \dfrac{R \sin \phi - a}{\sin \phi}} = \sin \phi = \frac{r \sin \phi}{r \sin \phi + R \sin \phi - a}$$

Dividing by $\sin \phi$ and clearing of fractions, $r = r \sin \phi + R \sin \phi - a$; transposing, $r (1 - \sin \phi) = R \sin \phi - a$; whence $r = \frac{R \sin \phi - a}{1 - \sin \phi}$, which is the formula sought. Substituting in this formula, the values given,

$$r = \frac{2 \sin 7 \text{ deg. } 30 \text{ min.} - 0.21875}{1 - \sin 7 \text{ deg. } 30 \text{ min.}}$$

$$= \frac{2 \times 0.130526 - 0.21875}{1 - 0.130526} = 0.04865 \text{ inch}$$

Second Solution: The formula for finding the radius of the smaller circle can also be derived as follows: Given EG (Fig. 23) perpendicular to AG, segment $FG = a$, $AD = R$, and the angle ϕ; to find r. $GE = a + r$, and $AE = R + r$. Then the triangle AEG is a right triangle, and $GE = EA \sin \phi$. Substituting the values for GE and AE found above:

$$a + r = (R + r) \sin \phi$$
$$= R \sin \phi + r \sin \phi$$
$$r (1 - \sin \phi)$$
$$= R \sin \phi - a$$
$$r = \frac{R \sin \phi - a}{1 - \sin \phi}$$

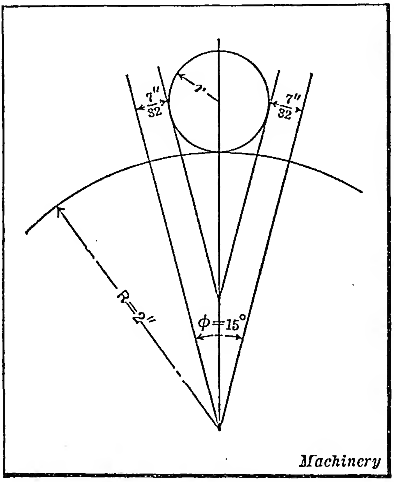

Fig. 24. Diagram for Third Solution of Problem represented by Fig. 22

Third Solution: In Fig. 24, as $R + r$ = distance between the centers of the circles,

$$\frac{r + 7/32}{R + r} = \sin \tfrac{1}{2} \phi = \sin 7 \text{ degrees } 30 \text{ minutes} = 0.130526$$

As $$R = 2 \; \frac{r + 0.21875}{2 + r} = 0.130526$$

and $$r + 0.21875 = 0.130526 (2 + r)$$
$$r - 0.130526r = 0.261052 - 0.21875$$

therefore $$0.869474r = 0.042302$$

and $$r = 0.042302 \div 0.869474 = 0.04865 \text{ inch}$$

To Find Dimensions x, y and Radius r, Fig. 25. The problem is to determine the center distance x, the height y and the radius r from the dimensions given.

Solution: There appear to be three unknowns, but there are only two in reality, since y can be found immediately as soon as r is known. Draw the vertical line AC and the horizontal lines BC and DE; join A and B and A and D. In the right triangles ACB and AED, $BC = x$, $AC = y - 0.256$, $AB = r + 0.256$, $DE = 0.973 - x$, $AE = y - 0.261$, and $AD = r + 0.015$. But $y = r + 0.204$; hence, $AC = r + 0.204 - 0.256 = r - 0.052$, and $AE =$

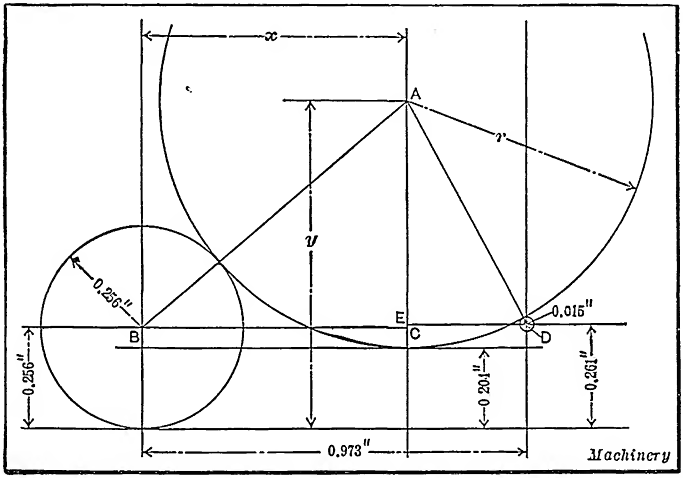

Fig. 25. To find Radius r and Dimensions x and y

$r + 0.204 - 0.261 = r - 0.057$. Since $BC^2 + AC^2 = AB^2$, $x^2 + (r - 0.052)^2 = (r + 0.256)^2$; whence, $x^2 + r^2 - 0.104r + 0.002704 = r^2 + 0.512r + 0.065536$, and

$$x^2 - 0.616r = 0.062832 \tag{1}$$

Also, since $DE^2 + AE^2 = AD^2$, $(0.973 - x)^2 + (r - 0.057)^2 = (r + 0.015)^2$; whence, $0.946729 - 1.946x + x^2 + r^2 - 0.114r + 0.003249 = r^2 + 0.03r + 0.000225$, and

$$x^2 - 1.946x - 0.144r = -0.949753 \tag{2}$$

Multiplying Equation (2) by 77 and Equation (1) by 18, we have:

$$77x^2 - 149.842x - 11.088r = -73.130981 \tag{3}$$

7 P

$$18x^2 - 11.088r = 1.130976 \tag{4}$$

Subtracting Equation (4) from Equation (3),

$$59x^2 - 149.842x = -74.261957$$

from which $x = 0.675001$ inch, or 1.86469 inch. Evidently, the smaller value is the one desired in this case. Substituting this value of x in Equation (1),

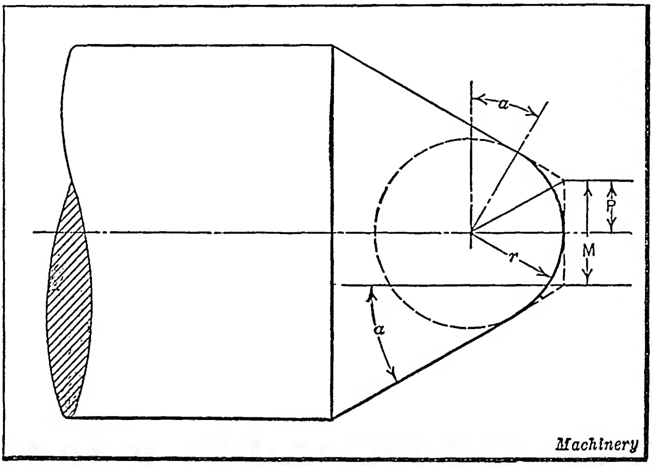

Fig. 26. To find Original Diameter M for Semispherical Shape at End of Conical Surface

$$0.675001^2 - 0.616r = 0.062832$$

from which

$$r = 0.63766 \text{ inch}$$

Finally,

$$y = 0.63766 + 0.204 = 0.84166 \text{ inch}$$

To Determine Diameter at End of Tapered Rod. When the tapered end of a rod has been machined to a semi-spherical form, as shown in Fig. 26, it is sometimes desirable to determine the original diameter M at the end of the rod. In the illustration, the known dimensions are:

a = angle of taper at end of rod; and
r = radius at rounded end.

With these values known, it is required to find M, or in other words, to find the diameter at the end of the tapered rod before the

end is rounded to radius r. From the diagram it is evident that

$$P = r\left(\tan\frac{90 - a}{2}\right) \qquad \text{Hence, } M = 2r\left(\tan\frac{90 - a}{2}\right)$$

To Calculate Diameter at Intersection of Two Tapers. When two tapers are formed on a piece of work it is sometimes necessary

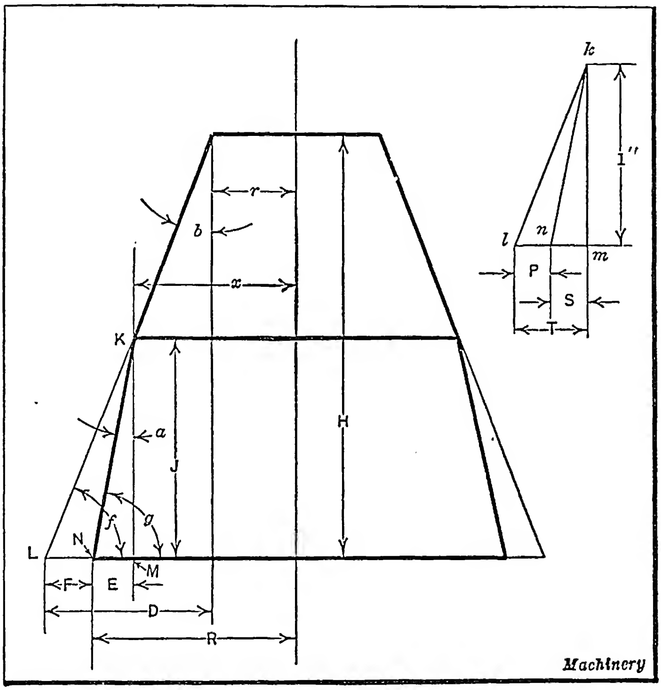

Fig. 27. To find Diameter at Intersection of Two Tapers

to determine the length of each taper and the diameter at the point where the two tapers join. This problem may involve the conditions shown in Fig. 27 or those shown in Fig. 28. In Fig. 27 it will be noticed that the section having the larger taper is at the top, while in Fig. 28 the section having the larger taper is at the bottom. Considering the problem as presented in Fig. 27, let

R = radius at large end of piece;
r = radius at small end of piece;

a = angle of taper at large end;
b = angle of taper at small end; and
H = height of piece.

Example 1: The problem is to find the length J of the tapered section at the large end, and the radius x at the intersection of the two tapers. From Fig. 27 we see that

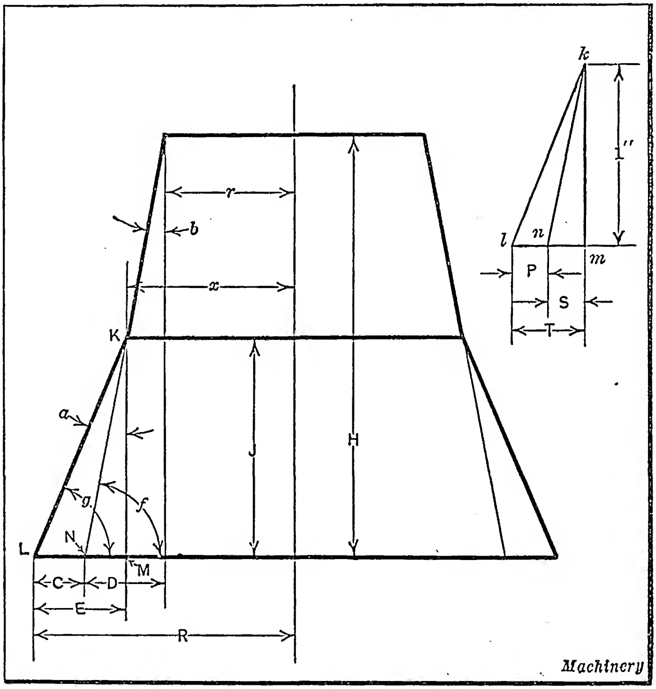

Fig. 28. Problem Similar in Principle to One represented by Fig. 27

$$D = H \tan b \qquad \text{and} \qquad F = D + r - R$$
$$f = 90 \text{ degrees} - b \qquad \text{and} \qquad g = 90 \text{ degrees} - a$$

In the small diagram in the upper right-hand corner, $km = 1$ inch. Now as angle lkm = angle b as shown in the large diagram, and angle nkm = angle a, it follows that the geometrical figure $LKMN$ is similar to the small geometrical figure $lkmn$. Therefore, according to trigonometry, we have

$$T = \cot f \qquad \text{and} \qquad S = \cot g$$

Therefore,

$$P = \cot f - \cot g$$

Now in the two similar geometrical figures we have

$$F : J :: P : 1$$

Thus,

$$PJ = F \qquad \text{and} \qquad J = \frac{F}{P} = \frac{D + r - R}{\cot f - \cot g}$$

By trigonometry $E = J \tan a$, and by subtraction $x = R - E$ and $2x$ = diameter at the intersection of the two tapers.

Example 2: In Fig. 28 illustrating the case in which the large taper is at the bottom end of the piece, let

R = radius at large end of piece;
r = radius at small end of piece;
a = angle of large taper;
b = angle of small taper; and
H = height of piece.

The problem is to find the height J of the tapered section at the large end, and the radius x at the intersection of the two tapers. In the illustration

$$D = H \tan b \qquad \text{and} \qquad C = R - D - r$$

Also,

$$f = 90 \text{ degrees} - b \qquad \text{and} \qquad g = 90 \text{ degrees} - a$$

In the small triangle in the upper right-hand corner, angle lkm = angle a in the larger figure, and angle nkm = angle b.

According to trigonometry

$$T = \cot g \qquad \text{and} \qquad S = \cot f$$

Therefore,

$$P = \cot g - \cot f$$

Considering the two similar geometrical figures, we have

$$C : J :: P : 1$$

Thus,

$$PJ = C \qquad \text{and} \qquad J = \frac{C}{P} = \frac{R - D - r}{\cot g - \cot f}$$

Therefore,

$E = J \tan a$ and $x = R - E$, and $2x$ = the diameter at the intersection of the two tapered sections.

To Determine Radius x, Fig. 29. Three disks of different radii are in contact. The three radii indicated on the diagram are given. Find the radius x of the smallest disk.

Solution: From Fig. 29 we have the conditions shown in Fig. 30, from which the value of x can be found as follows:

In triangle ABC

$$\text{Cos } B = \frac{a^2 + c^2 - b^2}{2ac} \tag{1}$$

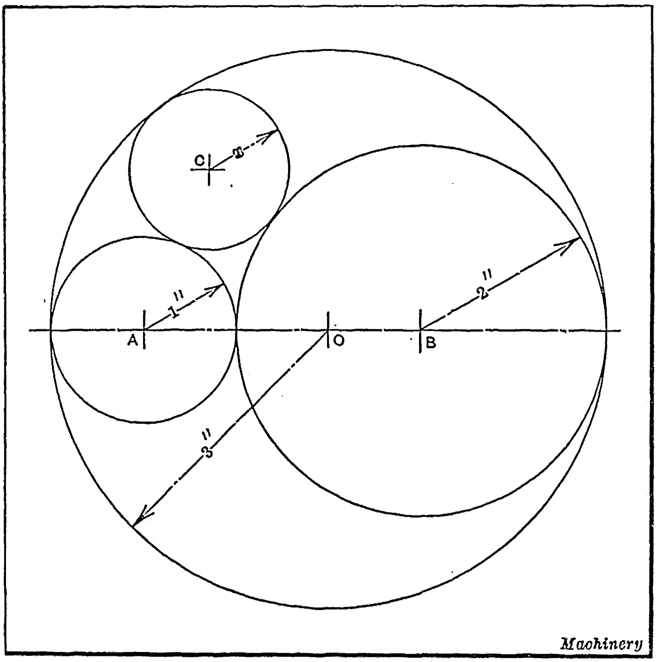

Fig. 29. To determine Radius x of Small Circle

In triangle OBC

$$\text{Cos } B = \frac{a^2 + m^2 - h^2}{2am} \tag{2}$$

From (1) and (2) by comparison

$$m\,(a^2 + c^2 - b^2) = c\,(a^2 + m^2 - h^2)$$

or

$$m\,[a^2 + (c + b)\,(c - b)] = c\,[a^2 + (m + h)\,(m - h)] \tag{3}$$

Let $a = 2 + x$, $b = 1 + x$, $c = 3$, $h = 3 - x$, and $m = 1$.

Then

$$1\,[(2 + x)^2 + (4 + x)\,(2 - x)] =$$
$$3\,[(2 + x)^2 + (4 - x)\,(x - 2)]$$
$$12 + 2x = 3\,(10x - 4) \text{ or } 28x = 24$$

$$x = \frac{6}{7} = 0.857 \text{ inch}$$

Diameter of Circumscribed Circle Tangent to Three Smaller Circles. The diagram, Fig. 31, represents a circumscribed circle which is tangent to three smaller circles. The problem is to deter-

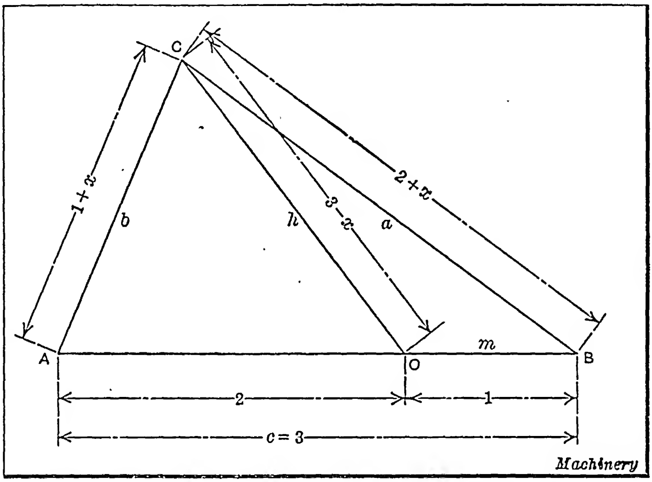

Fig. 30. Diagram illustrating Method of finding Radius x, Fig. 29

mine the radius of the circumscribed circle. For example, if the diameters of three circles A, B, and C are 2, $1\frac{7}{16}$, and $1\frac{1}{16}$ inches, what is the smallest circle that will include them?

Solution: The radius of this circle can be found by the following formula:

$$R = \frac{abc}{2\sqrt{abc\,(a + b + c)} - (ab + bc + ac)}$$

in which

R = radius of circumscribed circle;
a = radius of circle A;
b = radius of circle B;
c = radius of circle C.

If circle A is 2 inches in diameter; B, $1\frac{7}{16}$ inches; and C, $1\frac{1}{16}$ inches; then $a = 1$ inch; $b = \frac{23}{32}$ or 0.71875 inch; and $c = \frac{17}{32}$ or 0.53125 inch. Substituting these values in the above formula the following is obtained:

$$R = \frac{0.38184}{1.85378 - 1.63184} = 1.720 \text{ inches}$$

Therefore the diameter of the circumscribed circle is $2 \times 1.720 = 3.440$ inches.

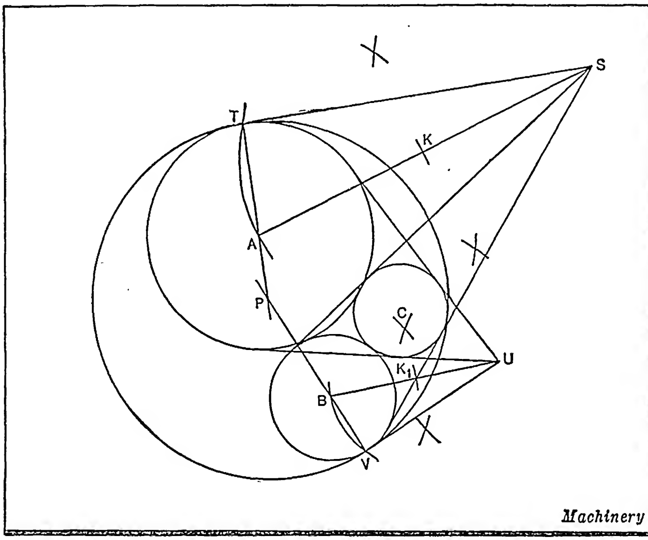

Fig. 31. To find Diameter of Circumscribed Circle Tangent to Three Smaller Circles

The center of the circumscribed circle is found in the following manner: Draw two lines tangent to circles B and C which intersect at S, their center of similitude. Draw TS tangent to circle A, and at the point of tangency erect a perpendicular, which will pass through the center of circle A. Next draw two lines tangent to circles A and C, which will intersect at U, their center of similitude, and draw line VU tangent to circle B. From this point of tangency also erect a perpendicular, which will pass through the center of circle B. The point of intersection of this perpendicular and the line perpendicular to TS will be the center, P, of the circumscribed circle.

As accuracy is required in laying out this diagram, in order to locate the center of the circumscribed circle exactly, perpendicular *TP* should be erected in the following manner, which eliminates the necessity of drawing *TS* and reduces the chance of errors. Draw *AS* from the center of similitude to the center of the circle and bisect

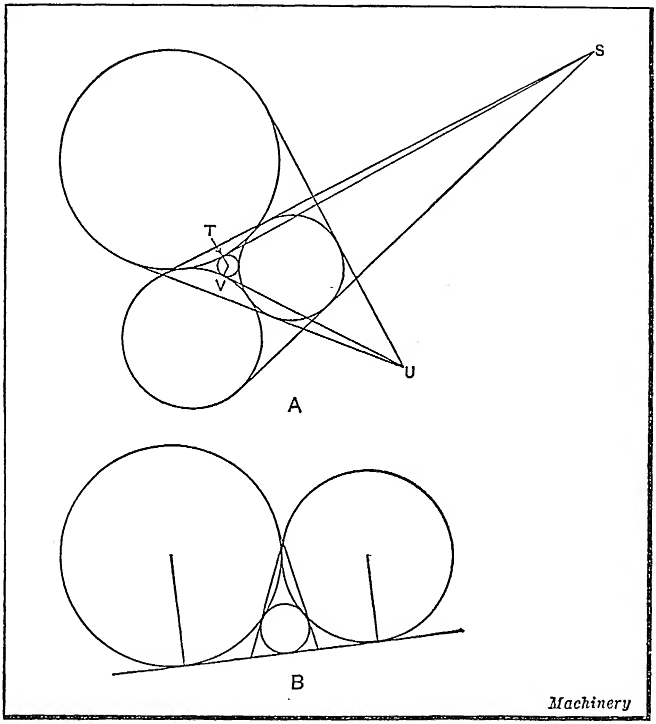

Fig. 32. Method of drawing Inscribed Circle Tangent to Three Circles

this line at *K* as shown. Then with *KA* as a radius, draw an arc intersecting the circumference of the circle at *T*, after which line *TP* can be drawn by intersecting point *T* and center *A*. The same method should also be used in drawing perpendicular *VP*.

If the tangent lines *TS* and *VU* had been drawn to the opposite points of tangency, as shown at *A* in Fig. 32, the intersection of the two perpendiculars erected from this point of tangency would have

been the center of an inscribed circle tangent to the three circles. In cases where the line tangent to three circles is a straight line, the perpendicular lines erected by the first method will not intersect but will be parallel as shown at B, Fig. 32. When one of the circles is so small that it is within a line tangent to the other two circles, as shown in Fig. 33, the perpendiculars erected by the first method described give the center of a circle which is tangent to the three

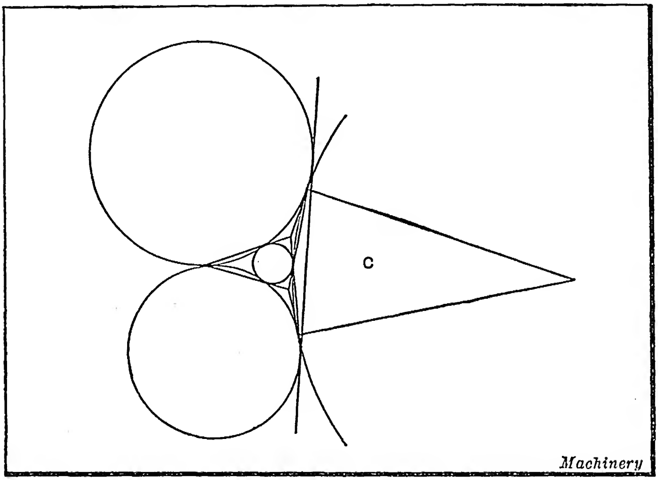

Fig. 33. Case in which Smallest of Three Tangent Circles is within a Line Tangent to the Other Two Circles

circles but which does not circumscribe the circles. The formula for finding the radius of such a tangential circle is as follows:

$$R = \frac{abc}{ab + bc + ac - 2\sqrt{abc\,(a + b + c)}}$$

in which the various letters represent the same elements as in the previous formula.

Diameter of Circle Tangent to Three Outer Circles. The problem is to find the diameter of disk X in Fig. 34, when the diameters of the other disks and the distances between their respective centers are known. This is the problem of four tangent circles. It is a special case in which the diameters of two of the circles are equal, but it can be solved by the general solution which is as follows:

Solution: In Fig. 35, let L, M, and N be the centers of the three given circles whose radii are a, b, and c, respectively, and let $MN = x$, $LN = y$ and $LM = z$; then find r. An equation of the second degree can be obtained as follows:

Let $LO = p$, $MO = q$, $NO = s$, and angle $LOM = \theta$, angle $MON = \phi$, and angle $NOL = \alpha$.

Then

$$\theta + \phi + \alpha = 360 \text{ degrees}$$
$$\theta = 360 \text{ degrees} - (\phi + \alpha)$$
$$\text{Cos } \theta = \cos(\phi + \alpha) = \cos\phi\cos\alpha - \sin\phi\sin\alpha$$
$$\text{Cos } \theta - \cos\phi\cos\alpha = -\sin\phi\sin\alpha$$

Squaring both sides,

$$\begin{aligned} &\text{Cos}^2\theta - 2\cos\theta\cos\phi\cos\alpha + \cos^2\phi\cos^2\alpha \\ &= \sin^2\phi\sin^2\alpha \\ &= (1 - \cos^2\phi)(1 - \cos^2\alpha) \\ &= 1 - \cos^2\phi - \cos^2\alpha + \cos^2\phi\cos^2\alpha \end{aligned}$$

Hence

$$\text{Cos}^2\theta + \cos^2\phi + \cos^2\alpha - 2\cos\theta\cos\phi\cos\alpha = 1 \qquad (1)$$

From the cosine formulas in trigonometry:

$$\text{Cos } \theta = \frac{p^2 + q^2 - z^2}{2pq} = 1 + \frac{(p - q)^2 - z^2}{2pq}$$
$$\text{Cos } \phi = \frac{q^2 + s^2 - x^2}{2qs} = 1 + \frac{(q - s)^2 - x^2}{2qs}$$
$$\text{Cos } \alpha = \frac{s^2 + p^2 - y^2}{2sp} = 1 + \frac{(s - p)^2 - y^2}{2sp}$$

Put, in the numerators, $p = r + a$, $q = r + b$, and $s = r + c$, and let A, B, and C represent certain expressions in the results; then,

$$1 + \frac{(p - q)^2 - z^2}{2pq} = 1 + \frac{(a - b)^2 - z^2}{2pq} = 1 + \frac{A}{2pq}$$
$$1 + \frac{(q - s)^2 - x^2}{2qs} = 1 + \frac{(b - c)^2 - x^2}{2qs} = 1 + \frac{B}{2qs}$$
$$1 + \frac{(s - p)^2 - y^2}{2sp} = 1 + \frac{(c - a)^2 - y^2}{2sp} = 1 + \frac{C}{2sp}$$

By substituting these values in Equation (1), performing the operations indicated, clearing of fractions and cancelling, the following equation is obtained:

$$A^2s^2 + B^2p^2 + C^2q^2 - 2ABsp - 2BCpq - 2CAqs - ABC = 0 \qquad (2)$$

Putting $p = r + a$, $q = r + b$, and $s = r + c$, in Equation (2), performing the operations indicated, and combining,

$$r^2 [A^2 + B^2 + C^2 - 2AB - 2BC - 2CA] + 2r [A^2c + B^2a + C^2b - AB (a + c) - BC (a + b) - CA (b + c)] + [A^2c^2 + B^2a^2 + C^2b^2 - 2ABca - 2BCab - 2CAbc - ABC] = 0 \quad (3)$$

This is a general quadratic equation, from which two values for r are always found; one applies to the circle making internal contact

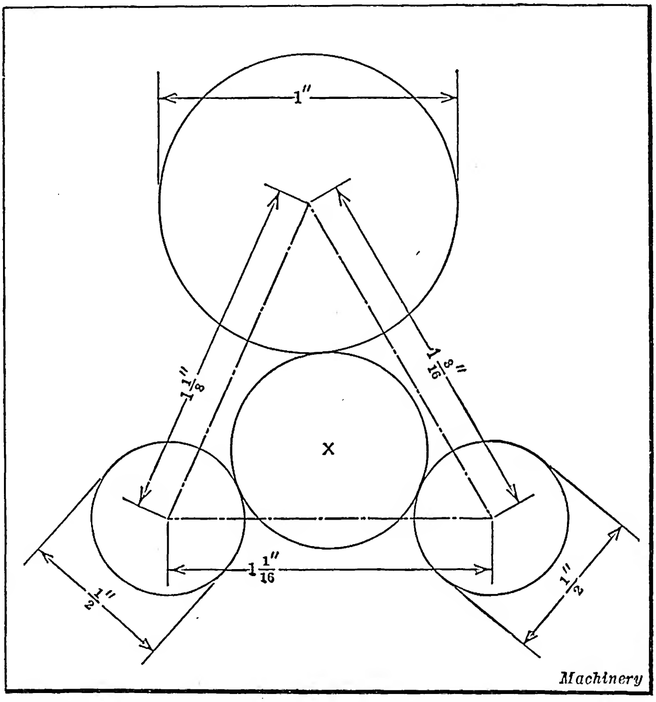

Fig. 34. To find Diameter of Circle X Tangent to Three Outer Circles

with all three circles, and the other to the circle making external contact with all three circles.

From Fig. 34, $a = \frac{1}{2}$ inch, $b = \frac{1}{4}$ inch, $c = \frac{1}{4}$ inch, $x = 1\frac{1}{16}$ inches, $y = 1\frac{3}{16}$ inches, and $z = 1\frac{1}{8}$ inches.

For convenience, reduce all values to sixteenths and use only the numerators in carrying out the calculations; this is the same as multiplying the roots of Equation (3) by 16. Then, $a = 8$, $b = 4$, $c = 4$, $x = 17$, $y = 19$, and $z = 18$; and

$$A = (a - b)^2 - z^2 = -308$$
$$B = (b - c)^2 - x^2 = -289$$
$$C = (c - a)^2 - y^2 = -345$$

Inserting these values in Equation (3), and solving, the different coefficients can be found as follows:

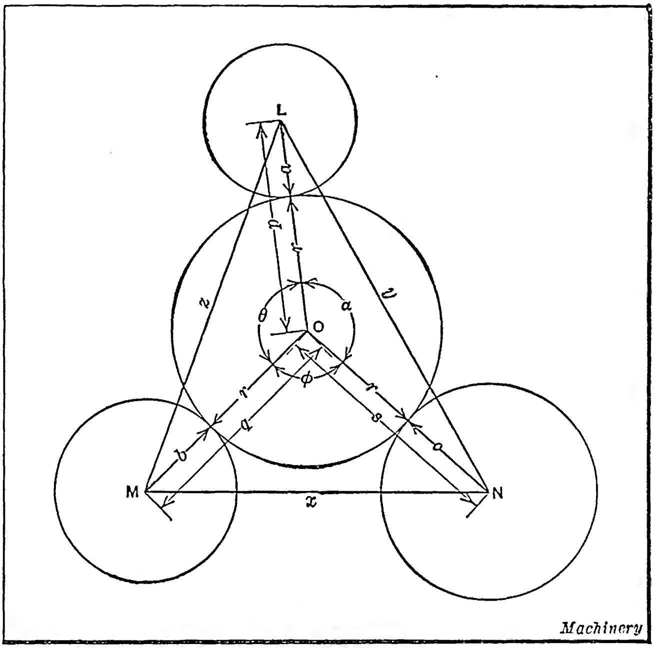

Fig. 35. Diagram for Solution of Problem represented by Fig. 34

$$\text{Coefficient of } r^2 = A^2 + B^2 + C^2 - 2AB - 2BC - 2CA$$
$$= -292{,}544$$

$$\text{Coefficient of } 2r = A^2c + B^2a + C^2b - AB\,(a + c) - BC\,(a + b) - CA\,(b + c)$$
$$= -1{,}590{,}960$$

$$\text{Absolute term} = A^2c^2 + B^2a^2 + C^2b^2 - 2ABca - 2BCab - 2CAbc - ABC = +23{,}998{,}500$$

This gives the quadratic equation,

$$292{,}544r^2 + 2\,(1{,}590{,}960)r - 23{,}998{,}500 = 0$$

Dividing by 4, this equation reduces to:

$$73{,}136r^2 + 2\,(397{,}740)r - 5{,}999{,}625 = 0$$

Solving for r,

$$r = \frac{-\,397{,}740 \pm \sqrt{397{,}740^2 + (73{,}136 \times 5{,}999{,}625)}}{73{,}136}$$

Using this equation with the plus sign before the radical to find the radius of a circle making external contact with all three circles,

$$r = 5.1262$$

Since the roots of the equation were multiplied by 16, divide the value of r by this number. Thus, $r = 0.3204$. Therefore, the radius of the disk X is 0.3204 inch, and the diameter is twice that or 0.6408 inch.

CHAPTER VI

GENERAL ENGINEERING AND DESIGNING PROBLEMS

In designing machine parts or any structural members which must withstand stresses, the method of procedure depends upon the function of the part, its requirements as to rigidity, and the nature and magnitude of the stresses which must be resisted. Frequently, parts are proportioned in accordance with a theoretical analysis of the stresses, but in many cases this is either impracticable or unnecessary. For instance, the dimensions of many machine details are determined by using empirical formulas containing constants which are intended to give dimensions proportional to similar designs which have been thoroughly tested in service. The use of such formulas is more practical in many instances than attempting to determine dimensions by a theoretical analysis; in fact, the latter method as applied to many parts would involve lengthy and complex calculations with greater chance of errors in obtaining the right proportions.

Another method of designing machine parts in certain other structural members is simply upon the basis of judgment and a knowledge of practical requirements, and without the use of formulas or mathematical calculations. This method is applied particularly to small and relatively unimportant details which do not require careful designing with reference to strength. There is still another general class of machine parts which must be designed to meet operating conditions instead of having the dimensions or proportions based upon strength. For instance, many machine tool parts, such as beds, frames and slides are made much more massive than is required merely for strength alone, because it is essential to have sufficient mass to prevent excessive vibrations when the machine, such as a grinder for example, is at work. There are also many other examples common to drafting-room practice, which are based partly upon strength calculations and partly upon the designer's judgment.

It is evident then, from what has been said about different methods of designing, that purely theoretical and mathematical processes,

as applied to machine parts, etc., have decided limitations. It is also apparent that judgment and experience play an important part in designing, since it is often impracticable to follow fixed rules, and it is of the utmost importance to know when and when not to design chiefly upon the basis of mathematical calculations. Competent draftsmen and machine designers should, of course, understand thoroughly the mathematical side of designing, and a variety of typical problems have been included in this chapter. These problems are not intended to form part of a logically arranged trea-

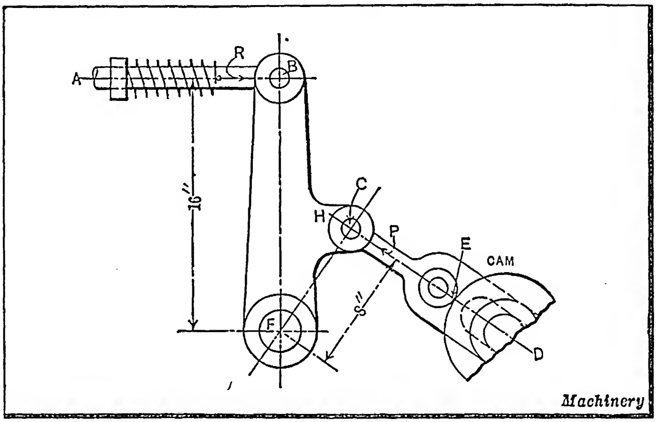

Fig. 1. Diagram illustrating Application of the Principle of Moments

tise, since the object is to present a diversified collection taken from a number of important branches of engineering work.

Application of the Principle of Moments. In Fig. 1 the lever shown is pivoted at F, which serves as the fulcrum. A push P is exerted by the rod at the right, which receives its motion from the cam and roller, as indicated. This push acts to overcome a resistance R which acts along the rod seen at the left, and which may be supposed to consist of the resistance of the spring coiled around the rod and of any piece of mechanism that this rod may have to operate. Let it be required to find how great a push P is necessary to overcome a resistance R of 250 pounds.

Solution: The first thing is to find the length of the true lever arms, since without these the moments cannot be determined. To do this, first draw lines through the points on the lever at which

the forces act, and in the direction in which they act. Thus, the force P acts at the point C, and the line DH indicates the position and direction of this force. Likewise the force R acts at point B, and line AB indicates the position and direction of force R.

Now, the lever arm of force P is the perpendicular distance from F to line DH, and the lever arm of force R is the perpendicular distance from F to line AB. Assume that these distances measure 8 and 16 inches, respectively. Then,

$$\text{Moment of } P = 8 \times P$$
$$\text{Moment of } R = 250 \times 16 = 4{,}000$$

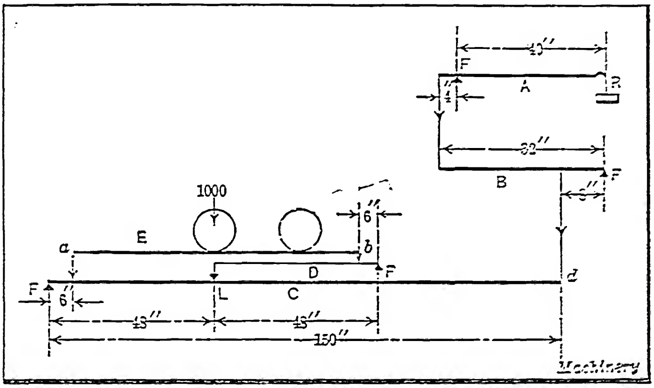

Fig. 2. Diagram representing Problem in Compound Levers

$$8 \times P = 4{,}000 \qquad \text{and} \qquad P = \frac{4{,}000}{8} = 500 \text{ pounds}$$

Calculation of Compound Levers. It often happens that it is necessary to use two or more levers connected one to the other in a series, where it would not be convenient to obtain the desired multiplication with a single lever, or where it is necessary to distribute the forces acting. The application of these "compound levers" is found in testing machines, car brakes, printing presses, and many other machines and devices. Probably the most familiar example is that of a pair of scales, and we will take this to illustrate the method of making the calculations for compound levers.

In Fig. 2 is a diagram showing an arrangement of levers that might be used for platform scales. The fulcrums of the various levers are in each case marked F. The scale platform is at E, bear-

ing at each end on levers C and D, and loaded at the center with 1,000 pounds. A pressure of 500 pounds, therefore, is transmitted to lever C at a point 6 inches from the fulcrum, and 500 to lever D. As lever D is proportioned exactly the same as that part of lever C to the left of the center line of the weight — that is, as the distance from F to L in each case is exactly 4 feet, and the short arms are each 6 inches long — it follows that the final effect is the same as though the whole 1,000 pounds acted at a point 6 inches from the fulcrum F of the lever C.

Continuing through the various connections, the right-hand end of C pulls down on the lever B at a point 8 inches from its fulcrum,

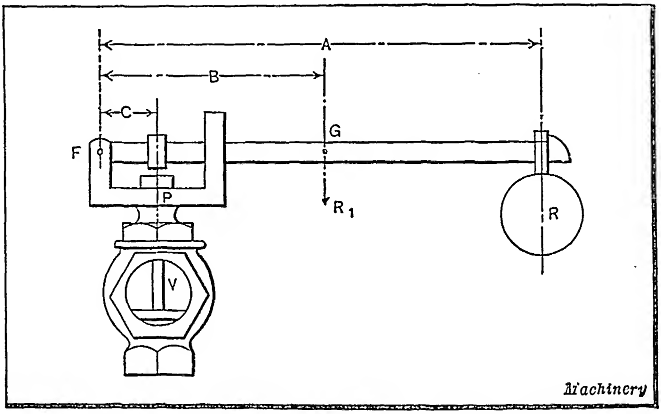

Fig. 3. Example of Lever Calculation with Three Forces to be considered

and lever B, in turn, pulls down on the scale beam A at a point 4 inches to the left of its fulcrum, and lifts the weight R.

Problem: What weight at R is required to balance the 1,000 pounds on the platform, assuming that the system of levers is in balance so that there is no unbalanced weight to be considered? This is always provided for by a counterpoise on the scale beam.

Solution: The best way to solve any example of compound levers is first to determine the number of multiplications of each lever. Lever A has arms 40 and 4 inches long, and multiplies 10 times; lever B multiplies 4 times; and lever C, 25 times. Each lever multiplies in the same direction; that is, it tends to increase the force acting when we start at point R. Hence, the total mul-

tiplication is $10 \times 4 \times 25 = 1000$, and thus one pound at R would balance the 1000 pounds on the platform.

It may be asked whether with this arrangement the weighing of the scale would not be altered should the weight be moved to the position shown dotted. A little thought will show that it would not. We have seen that the reduction from both points a and b to point d is 25 to 1, and it can make no difference whether 500 pounds acts at both a and b, or whether, for example, 300 pounds acts at a and 700 at b, the total 1000 pounds being reduced 25 to 1 in either case.

Safety Valve Calculations. The safety valve shown by the diagram, Fig. 3, is an example of a lever in which there are three forces to be considered, if we take into account the weight of the lever, which it is quite essential to do. The valve at V is acted upon by the pressure of the steam, tending to raise it. This pressure constitutes the push P upon the lever, which is resisted by the suspended weight R, and the weight of the lever, which we will call R_1. The weight of the lever is effective at the point G, the center of gravity of the lever. This point can be found by balancing the lever on a knife edge, the center of gravity being directly over the knife edge. The fulcrum of the lever is at F, and the lever arms for R, R_1 and P are marked A, B, and C, respectively.

Example 1: Assume that $A = 30$ inches, $B = 14$ inches, $C = 3$ inches, $R = 20$ pounds, and $R_1 = 8$ pounds. Find what pressure of steam the valve will carry.

$$\begin{aligned} \text{Moment of } P &= 3 \times P \\ \text{Moment of } R &= 20 \times 30 = 600 \\ \text{Moment of } R_1 &= 8 \times 14 = 112 \end{aligned}$$

For the valve to balance, the moment of P must be equal to the sum of the moments of R and R_1, for the moment of P tends to raise the lever, and the other moments tend to hold it down. Adding the moments of R and R_1, therefore, we have $600 + 112 = 712$, and this must balance the moment of P or $3 \times P$. Hence, $3 \times P = 712$, and $P = \dfrac{712}{3} = 237\frac{1}{3}$ pounds. The $237\frac{1}{3}$ pounds is the total pressure upon the valve, and to obtain the pressure per square inch that can be carried, we have simply to divide $237\frac{1}{3}$ by the area of the valve. To be theoretically exact, the weight of the valve and stem should be added to the figure $237\frac{1}{3}$.

Example 2: Suppose it were desired to carry a total pressure upon the valve of 300 pounds. With the other dimensions remain-

ing as before, how heavy a weight R would have to be provided? Again, taking moments, we have,

$$\begin{aligned} \text{Moment of } R &= 30 \times R \\ \text{Moment of } R_1 &= 8 \times 14 = 112 \\ \text{Moment of } P &= 300 \times 3 = 900 \end{aligned}$$

The sum of the first two moments must equal the last one, but we cannot add them as they stand, because we do not yet know what the first one is. Hence we will indicate the addition as follows:

$$30 \times R + 112 = 900$$

$$R = \frac{900 - 112}{30} = 26\tfrac{4}{15} \text{ pounds}$$

The following explanation will make the reason for subtracting 112 from 900 clear. We have found that the moment of R is 788;

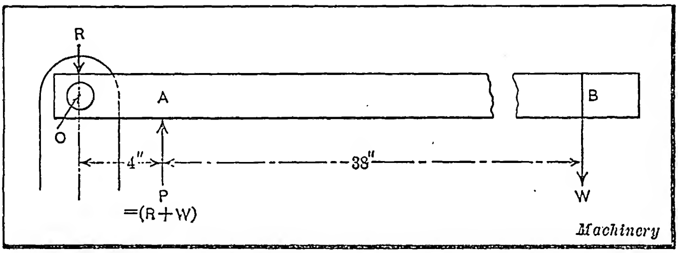

Fig. 4. Diagram for demonstrating why a Safety Valve is in Equilibrium

of R_1, 112; and of P, 900. Now, if 788 added to 112 equals 900, 900 must be 112 greater than 788, and 788 must be equal to 900 with 112 subtracted from it. Again, taking the formula as we have it, if $30 \times R$ plus 112 equals 900, $30 \times R$ must equal 900 with 112 subtracted from it.

Equilibrium of a Safety Valve. If a couple produces or tends to produce rotation only, and if it can be balanced only by another couple that tends to rotate the body in the opposite direction, how can this be reconciled with the forces acting on a safety-valve lever? Here the steam pressure acts upward and the weight acts downward; if this forms a couple, where is the other couple?

In Fig. 4, the force P acting upward represents the steam pressure, and the force W acting downward represents the weight. The lever would turn under the action of these two forces if it were not for the pin O. As the bar presses against the under side of this,

the pin reacts downward an equal amount; this force is represented by R. We may imagine that the pin O is removed and that equilibrium is maintained by means of a force R acting on top of the lever in a vertical line through the center of the pin. Then, force P must equal $R + W$, since it supports both W and R, both acting downward. Force P may, therefore, be considered as made up of two forces, one equal to R and the other equal to W, and both acting upward. This force R combined with the downward force R forms a couple, the arm of which is OA, and which tends to turn the bar counter-clockwise; W combined with the force (weight) W forms another and equal couple, the arm of which is AB, and which tends to turn the bar clockwise; the two couples thus neutralize each other.

Taking an actual case, suppose $W = 28$ pounds, $OA = 4$ inches, and $AB = 38$ inches. Then, taking the center of moments at A, $R \times 4 = 28 \times 38$, or $R = 266$ pounds. Taking the center of moments at O, $P \times 4 = 28 \times (38 + 4)$, or $P = 294$ pounds $= 266 + 28 = R + W$, as it should. The assumption of a couple formed by P and W is wrong, because a couple consists of two equal parallel and opposite forces, and P is greater than W. Note that the weight of the bar (lever) has been neglected.

Time Required for Body to Fall from Given Height. If a metal ball falls from the top of a tower 300 feet high, how long a time will be required before it reaches the ground?

The formula by means of which this problem is solved is:

$$t = \sqrt{\frac{2h}{g}}$$

in which t = time in seconds;
h = height in feet;
g = acceleration due to gravity = 32.16 feet.

Inserting the known values of h and g in the formula, we have:

$$t = \sqrt{\frac{2 \times 300}{32.16}} = \sqrt{18.66} = 4.32 \text{ seconds}$$

Velocity of Falling Body. What is the velocity of the ball in the previous example when it reaches the ground?

The formula for finding the velocity is:

$$v = \sqrt{2gh}$$

in which v = velocity in feet per second, and h and g denote the same quantities as in the preceding problem. Inserting the values of g and h in the formula, we have:

$$v = \sqrt{2 \times 32.16 \times 300} = \sqrt{19{,}296} = 139 \text{ feet, nearly}$$

Height Reached by Projectile. A projectile is fired from a 12-inch gun vertically into the air. It strikes the ground, coming down, exactly 1 minute and 40 seconds after it left the muzzle. Disregarding air resistance, what height did the projectile reach? What was its velocity when leaving the muzzle? And what is the energy of the projectile when it strikes the ground, if its weight is assumed to be 600 pounds?

The time required for the projectile to reach its greatest height is one-half of the total time for the upward and downward journey. Thus, in 50 seconds, the projectile has reached the point where its velocity is zero, and where it begins to fall. The formula for finding the height reached is:

$$h = \frac{gt^2}{2}$$

in which h, g and t denote the same quantities as in the two preceding examples. Inserting the known values, we have:

$$h = \frac{32.16 \times 50^2}{2} = \frac{32.16 \times 2{,}500}{2} = 40{,}200 \text{ feet, or}$$

$$\frac{40{,}200}{5{,}280} = 7.6 \text{ miles, approximately}$$

Muzzle Velocity, Energy, and Force of Blow. The velocity of the projectile when leaving the muzzle is the same as the velocity acquired when again reaching the ground. This velocity is found by the formula:

$$v = gt = 32.16 \times 50 = 1{,}608 \text{ feet per second}$$

The energy of the projectile when it strikes the ground equals its weight multiplied by the distance through which it has fallen. If W = weight, and E = energy, we have:

$$E = W \times h = 600 \times 40{,}200 = 24{,}120{,}000 \text{ foot-pounds}$$

Another formula for the energy is as follows:

$$E = \frac{Wv^2}{2g}$$

This formula, with the values of W, v and g inserted, will, of course, give the same result.

$$E = \frac{600 \times 1608^2}{2 \times 32.16} = \frac{600 \times 2{,}585{,}664}{2 \times 32.16} = 24{,}120{,}000 \text{ foot-pounds}$$

If, upon reaching the ground, the projectile buries itself to a depth of 8 feet, what is the average force of the blow with which it

strikes the ground? The average force of the blow equals the energy divided by the distance d in which it is used up, plus the weight of the projectile, or if F = average force of blow:

$$F = \frac{E}{d} + W = \frac{24{,}120{,}000}{8} + 600 = 3{,}015{,}600 \text{ pounds}$$

Kinetic Energy of Drop-hammer and Average Force of Blow. A drop-hammer weighing 300 pounds falls through a distance of 3 feet. What is the stored or kinetic energy of the hammer when it strikes the work, and what is the average force with which it delivers the blow, if, on striking the work, it compresses it ½ inch?

From the formula for finding foot-pounds given in the preceding problem, we have:

$$E = W \times h = 300 \times 3 = 900 \text{ foot-pounds}$$

The distance d in which this energy is used up equals ½ inch or ½ ÷ 12 = 0.04 foot. Therefore, the average force is:

$$F = \frac{E}{d} + W = \frac{900}{0.04} + 300 = 22{,}500 + 300 = 22{,}800 \text{ pounds}$$

Stress in Flywheel Rim. Find the stress in the rim of a flywheel, 5 feet mean diameter, made of cast iron, the rim being 2 inches wide by 4 inches deep, if the flywheel rotates at a velocity of 200 revolutions per minute.

The formula for the stress in the rim is:

$$S = 0.00005427WRr^2$$

in which S = stress in pounds on the rim section;
W = weight of rim in pounds;
R = mean radius in feet;
r = revolutions per minute.

We know that the mean diameter of the flywheel is 5 feet; therefore, R = 2.5 feet; r is given as 200; but we must find the value of W before we can find the value of S.

The weight W of the rim equals its volume or content in cubic inches multiplied by the weight of cast iron per one cubic inch. The volume of the rim equals the cross-sectional area of the rim multiplied by the circumference of the circle having for radius the mean radius of the flywheel; expressed as a formula:

$$V = 2R \times 3.1416 \times a \times b$$

in which V = the volume of the rim, in cubic inches, R = the mean radius, in inches, a = the width, and b = the depth of the rim, in inches. Substituting the values in this formula, we have:

$$V = 2 \times 30 \times 3.1416 \times 2 \times 4 = 1508 \text{ cubic inches}$$

One cubic inch of cast iron weighs 0.26 pound. The weight of the rim then is:

$$W = 1508 \times 0.26 = 392 \text{ pounds}$$

We can now substitute the values in the formula for determining the value of S.

$$S = 0.00005427 \times 392 \times 2.5 \times 200^2 = 2{,}127 \text{ pounds}$$

The multiplication above can be carried out by the use of logarithms as follows:

$$\begin{aligned} \log 0.00005427 &= \bar{5}.73456 \\ \log 392 &= 2.59329 \\ \log 2.5 &= 0.39794 \\ 2 \times \log 200 &= 4.60206 \\ \hline \log S &= 3.32785 \end{aligned}$$

Hence $S = 2{,}127$ pounds.

Mean Effective Pressure in Engine Cylinder. The initial absolute pressure of the steam in a steam engine cylinder is 120 pounds; the length of the stroke is 26 inches, the clearance $1\frac{1}{2}$ inches, and the period of admission, measured from the beginning of the stroke, 8 inches. Find the mean effective pressure.

The mean effective pressure is found by the formula:

$$p = \frac{P\,(1 + \text{hyp. log } R)}{R}$$

in which p = mean effective pressure in pounds per square inch;
P = initial absolute pressure in pounds per square inch;
R = ratio of expansion, which in turn is found from the formula:

$$R = \frac{L + C}{l + C}$$

in which L = length of stroke in inches;
l = period of admission in inches;
C = clearance in inches.

The given values are $P = 120$; $L = 26$; $l = 8$; and $C = 1\frac{1}{2}$. By inserting the last three values in the formula for R, we have:

$$R = \frac{26 + 1\frac{1}{2}}{8 + 1\frac{1}{2}} = \frac{27.5}{9.5} = 2.89$$

If we now insert the value of P and the found value of R in the formula for p, we have:

$$p = \frac{120\,(1 + \text{hyp. log } 2.89)}{2.89}$$

The hyperbolic logarithm (hyp. log.) must be found from tables giving its value, such as are found in MACHINERY'S HANDBOOK. The hyperbolic logarithm for 2.89 is 1.0613. Inserting this value in our formula, we have:

$$p = \frac{120\,(1 + 1.0613)}{2.89} = \frac{120 \times 2.0613}{2.89} = 85.6 \text{ lbs. per square inch}$$

To Calculate Indicated Horsepower of Steam Engine. The cylinder of a steam engine is 16 inches in diameter, and the length of the piston stroke 20 inches. The mean effective pressure of the steam on the piston is 110 pounds per square inch, and the number of revolutions per minute of the engine flywheel is 80. What is the power of the engine in indicated horsepower?

The formula for the horsepower (H.P.) of an engine follows:

$$\text{H.P.} = \frac{PLAN}{33{,}000}$$

in which P = mean effective pressure in pounds per square inch;
L = length of stroke in feet;
A = area of piston in square inches;
N = number of strokes of piston per minute.

In the given problem $P = 110$; L (in feet) $= \frac{20}{12} = 1\frac{2}{3}$; A, the area of the piston in square inches $= 16^2 \times 0.7854 = 256 \times 0.7854 = 201.06$; and N, the number of strokes of piston per minute $= 2 \times$ revolutions of flywheel $= 2 \times 80 = 160$. Substituting these values in the formula, we have:

$$\text{H.P.} = \frac{110 \times 1\frac{2}{3} \times 201.06 \times 160}{33{,}000} = 178.72$$

Cylinder Diameter and Stroke for Given Number of Horsepower. It is required to determine the diameter of cylinder and length of stroke of a steam engine to deliver 150 horsepower. The mean steam pressure is 75 pounds; the number of strokes per minute is 120. The length of the stroke is to be 1.4 times the diameter of the cylinder.

First insert in the horsepower formula the known values:

$$150 = \frac{75 \times L \times A \times 120}{33{,}000} = \frac{3 \times L \times A}{11}$$

The last expression is found by cancellation.

Assume now that the diameter of the cylinder in inches equals D.

Then $L = \frac{1.4D}{12} = 0.117\ D$, according to the requirements in the problem; the divisor 12 is introduced to change the inches to feet, L being in feet in the horsepower formula. The area $A = D^2 \times 0.7854$. If we insert these values in the last expression in our formula, we have:

$$150 = \frac{3 \times 0.117\ D \times 0.7854\ D^2}{11} = \frac{0.2757\ D^3}{11}.$$

$$0.2757\ D^3 = 150 \times 11 = 1650$$

$$D^3 = \frac{1{,}650}{0.2757} \qquad D = \sqrt[3]{\frac{1650}{0.2757}} = \sqrt[3]{5984.8} = 18.15$$

The diameter of the cylinder, thus, should be about $18\frac{1}{4}$ inches, and the length of the stroke $18.15 \times 1.4 = 25.41$, or, say, $25\frac{1}{2}$ inches.

Conversion of Torque into Horsepower. When applying power by rotation, the pull required on a pulley varies inversely as the radius at which the pull is applied, and when power is expressed as "torque" it generally means the pull required at 1 foot radius. Thus the torque of a motor is its turning moment or the number of pounds of effort exerted at a radius of 1 foot.

To find the horsepower that is equivalent to a given torque and speed of rotation, the following formula may be used, in which

H = horsepower;
T = torque at 1 foot radius; and
R = revolutions per minute.

$$H = \frac{T \times R \times 2 \times 3.1416}{33{,}000} = \frac{T \times R}{5252} \qquad T = \frac{H \times 5252}{R}$$

Horsepower-hour Defined. Power is the rate of doing work, and time is always considered. The unit of power is 1 foot-pound of work performed in 1 second. If the resistance in pounds be represented by r, the distance (space) in feet through which the resistance is overcome by s, the time in seconds during which the work is done by t, and the number of power units developed by p, then

$$p = \frac{r \times s}{t}$$

One horsepower is 550 power units; hence, dividing both sides of the equation by 550,

$$\frac{p}{550} = \text{H.P.} = \frac{rs}{550\,t}$$

Dividing both terms of the fraction by 60,

$$\text{H.P.} = \frac{rs \div 60}{550\ (t \div 60)}$$

But the time in seconds divided by 60 is the time in minutes, and letting $t \div 60 = T$,

$$\text{H.P.} = \frac{rs}{60 \times 550\ T} = \frac{rs}{33{,}000\ T}$$

Dividing both terms of the last fraction by 60,

$$\text{H.P.} = \frac{rs \div 60}{33{,}000\ (T \div 60)}$$

But the time in minutes divided by 60 is the time in hours; hence, letting $T \div 60 = T_1$,

$$\text{H.P.} = \frac{rs.}{60 \times 33{,}000\ T_1} = \frac{rs}{1{,}980{,}000\ T_1}$$

In other words, when $T_1 = 1$ hour, the product rs must equal 1,980,000 for H.P. to equal 1; and since rs is the number of foot-pounds of work performed, a horsepower-hour is 1,980,000 foot-pounds of work performed in one hour.

Number of B. T. U. in Horsepower-hour. A horsepower-hour is said to be equal to 2545 B.T.U. How is this number determined?

If a machine having a capacity of one horsepower were to run for one hour, it would perform 1,980,000 foot-pounds of work. Since one B.T.U. is equal to 778 foot-pounds of work, one horsepower-hour $= \frac{1{,}980{,}000}{778} = 2545$ B.T.U., very nearly. It should be noted that a horsepower-hour is not a unit of power, but a unit of work, and is equal to 1,980,000 foot-pounds. Similarly, 200 horsepower-hours may mean the work done by a 200-horsepower machine in one hour, or by a 25-horsepower machine in eight hours, etc., but in any case, it would equal $1{,}980{,}000 \times 200 = 396{,}000{,}000$ foot-pounds $= 2545 \times 200 = 509{,}000$ B.T.U.

Shaft Diameter for Transmitting Given Power. Find the diameter of shaft required to transmit 60 horsepower at 300 revolutions per minute, if the maximum safe stress of the material of which the shaft is made is 10,000 pounds per square inch.

The formula for finding the diameter of shaft is:

$$d = \sqrt[3]{\frac{321{,}400 \times \text{H.P.}}{RS}}$$

in which d = diameter of shaft in inches;
H.P. = horsepower to be transmitted;
R = revolutions per minute;
S = safe shearing stress of material of which shaft is made.

If we insert the given values in the given formula, we have:

$$d = \sqrt[3]{\frac{321{,}400 \times 60}{300 \times 10{,}000}} = \sqrt[3]{6.428} = 1.86 \text{ inch}$$

The diameter of the shaft may, therefore, be made, say, 1⅞ inches.

Length of a Plain Bearing. Determine the length of the main bearing of a large horizontal steam engine. The diameter of the crankshaft is 10 inches, and the weight of the shaft, flywheel, crankpin and other moving parts that may be supported by the bearings is 60,000 pounds. Assume that two thirds of this weight, or 40,000 pounds, comes on the main bearing. The engine runs at 80 revolutions per minute.

The length of the main bearing of an engine may be found by the formula:

$$L = \frac{W}{PK}\left(N + \frac{K}{D}\right)$$

in which L = length of bearing in inches;
W = load on bearing in pounds;
P = maximum safe unit pressure on bearing at a very slow speed;
K = constant depending upon the method of oiling and care which the journal is likely to get;
N = number of revolutions per minute;
D = diameter of bearing in inches.

The safe unit pressure P for shaft bearings is 400 pounds; the factor K varies from 700 to 2,000. In this case, assume first-class care and drop-feed lubrication, in which case $K = 1000$. The other quantities given are $W = 40{,}000$, $N = 80$, and $D = 10$.

Inserting these values in the formula for L we have:

$$L = \frac{40{,}000}{400 \times 1000}\left(80 + \frac{1000}{10}\right) = \frac{1}{10}(80 + 100) = 18 \text{ inches}$$

Force Required for Lifting Weight by Means of Screw. It is required to lift a weight weighing 1 ton by means of a screw having a lead of ½ inch. A lever passing through the head of the screw, and extending 4 feet out from the center, is provided at its outer end

with a handle. How great a force must be applied at this handle to lift the required weight, friction being disregarded?

Let the weight to be lifted, in pounds, be W; the force applied at the end of the lever, F; the lead of the screw, l; and the length of the lever, in inches, r. The distance passed through by force F times this force must equal the distance weight W is lifted times the weight, or, expressed as a formula:

$$F \times 2r \times 3.1416 = W \times l$$

This formula is based on the fact that during one revolution of the screw and handle, force F acts through a distance equal to the circumference of the circle described by the handle, while the weight W is lifted an amount equal to the lead of the screw. If we insert the given values in the formula, we have:

$$F \times 2 \times 48 \times 3.1416 = 2000 \times \tfrac{1}{2}$$

$$F \times 301.59 = 1000$$

$$F = \frac{1000}{301.59} = 3.3 \text{ pounds}$$

It will be seen that by the given arrangement a force of 3.3 pounds would be sufficient to lift a ton. Friction, however, has not been considered in this problem, and as the frictional resistance in machines using screws for conveying power is considerable, the actual force required would be a great deal more than 3.3 pounds.

Assume that it is required to find the power if friction is considered. In this case we must know the diameter of the screw and the form of the thread. We will assume that the thread is square, and that the diameter of the screw is 3 inches. The depth of a ½-inch lead square thread is ¼ inch. The pitch diameter of the screw is, therefore, $3 - \tfrac{1}{4} = 2\tfrac{3}{4}$ inches.

The formula for finding the force required at the end of the handle is:

$$Q = W \frac{f + \tan \alpha}{1 - f \tan \alpha} \times \frac{R}{r}$$

in which Q = force at end of handle, in pounds;
W = weight to be lifted = 2,000 pounds;
f = coefficient of friction;
α = angle of helix of the thread at the pitch diameter;
R = pitch radius of screw in inches = $1\tfrac{3}{8}$ inches;
r = length of handle in inches = 48.

$$\tan \alpha = \frac{\text{lead}}{3.1416 \times \text{pitch diam.}} = \frac{0.5}{3.1416 \times 2.75} = 0.058$$

The coefficient of friction, f, may be assumed to be 0.15. If we now insert the known values in the formula, we have:

$$Q = 2{,}000 \times \frac{0.15 + 0.058}{1 - 0.15 \times 0.058} \times \frac{1.375}{48} = 12.02 \text{ pounds}$$

or nearly four times as much as when friction was not considered.

Power Required for Compressing Given Volume of Air. Find the horsepower required for compressing 10 cubic feet of air per second from 1 to 12 atmospheres, including the work of expulsion from the cylinder. Frictional and other losses are disregarded.

The formula for the work, W, in foot-pounds, required for compression and expulsion of 1 cubic foot of air from p_1 to p_n atmospheres is:

$$W = 3.463\, p_1 \left[\left(\frac{p_n}{p_1}\right)^{0.29} - 1\right] \times 14.7 \times 144$$

In the given problem $p_1 = 1$; $p_n = 12$; and as we are to compress 10 cubic feet instead of one, we must multiply the whole expression by 10. Thus:

$$W = 3.463 \times 1 \times \left[\left(\frac{12}{1}\right)^{0.29} - 1\right] \times 14.7 \times 144 \times 10$$
$$= 3.463 \times (12^{0.29} - 1) \times 14.7 \times 144 \times 10$$

The value of the expression $12^{0.29}$ can be found only by the use of logarithms.

$$\log 12 = 1.07918$$
$$\log 12^{0.29} = 1.07918 \times 0.29 = 0.31296$$
$$12^{0.29} = 2.056, \text{ and } 12^{0.29} - 1 = 1.056$$

Hence:

$$W = 3.463 \times 1.056 \times 14.7 \times 144 \times 10 = 77{,}410 \text{ foot-pounds}$$

This last result may be found by ordinary multiplication, or, more quickly, by logarithms as follows:

$$\begin{aligned} \log\ 3.463 &= 0.53945 \\ \log\ 1.056 &= 0.02366 \\ \log\ 14.7 &= 1.16732 \\ \log\ 144 &= 2.15836 \\ \log\ 10 &= \underline{1.00000} \\ \log W &= 4.88879 \qquad W = 77{,}410 \end{aligned}$$

As a horsepower equals 550 foot-pounds per second, the horsepower required for compressing 10 cubic feet of air from 1 to 12 atmospheres equals:

$$\text{H.P.} = \frac{77{,}410}{550} = 151 \text{ horsepower}$$

Volume of Steam at a Given Pressure. Find the volume occupied by 6 pounds of dry and saturated steam at a gage pressure of 100 pounds per square inch, without the use of a steam table.

Solution: Rankine's formula follows:

$$PV^{\frac{17}{16}} = 475$$

where P = absolute pressure in pounds per square inch;
V = volume in cubic feet occupied by one pound of steam.

The absolute pressure = gage pressure + pressure indicated by the barometer; if the latter is not known, it is customary to call it 14.7. In this case we have:

$$P = 100 + 14.7 = 114.7$$
$$PV^{\frac{17}{16}} = 114.7V^{\frac{17}{16}} = 475$$

Taking the logarithm of both sides of this equation:

$$\text{Log } 114.7 + \frac{17}{16}\log V = \log 475$$

$$\text{Log } V = \frac{16}{17}(\log 475 - \log 114.7) = \frac{16}{17}(2.67669 - 2.05956) = 0.58083.$$

$V = 3.8092$ = number of cubic feet occupied by one pound of steam under the above conditions. The volume occupied by six pounds is:

$$3.8092 \times 6 = 22.8552, \text{ say, } 22.86 \text{ cubic feet}$$

The logarithm of any number (or quantity) having an exponent is equal to the logarithm of the number multiplied by the exponent; for instance, $\log A^c = c \times \log A$.

Weight of Steam Flowing through Pipe in Given Time. Find the weight of steam that will flow in one minute through a pipe 100 feet in length and 2 inches in diameter, if the initial pressure is 40 pounds (absolute) per square inch and the terminal or delivery pressure 35 pounds (absolute). The formula for finding the weight of steam under the given conditions is:

$$W = c\sqrt{\frac{w\,(P - P_1)\,d^5}{L}}$$

in which W = pounds of steam per minute;
c = constant = 52.7 for a 2-inch pipe;
w = weight per cubic foot of steam at initial pressure, in pounds;

P = initial pressure in pounds per square inch;
P_1 = terminal pressure in pounds per square inch;
d = diameter of pipe in inches;
L = length of pipe in feet.

In the present problem, $c = 52.7$; $w = 0.0972$ (obtained from tables in standard handbooks); $P = 40$; $P_1 = 35$; $d = 2$; and $L = 100$. Inserting these values in the formula gives:

$$W = 52.7\sqrt{\frac{0.0972 \times (40 - 35) \times 2^5}{100}} = 52.7\sqrt{0.1555} = 20.76$$
pounds

Discharge of Air into the Atmosphere. The following general formula may be used for determining the rate at which free air flows through an orifice of known size into the atmosphere, the gage pressure in the tank being anywhere from 5 to 50 pounds per square inch.

Let T = absolute temperature of air in tank in degrees F. = 460 + temperature indicated by thermometer; let p = absolute pressure in tank = gage pressure + pressure indicated by barometer in pounds per square inch; p_1 = pressure of atmosphere as indicated by barometer; and V = velocity of flow through orifice in feet per second. Then the theoretical velocity of discharge is:

$$V = 108.67\sqrt{T\left[1 - \left(\frac{p_1}{p}\right)^{0.2907}\right]}$$

If a barometer is not available, assume that $p_1 = 14.7$ and p = gage pressure + 14.7. The actual velocity will not be so great as calculated by the formula, because it will be affected by the size and shape of the orifice, practically the same conditions obtaining as in the case of the discharge of water. If the discharge is through a short tube the length of which is two or three times the diameter of the orifice, the actual velocity of discharge may be taken as $0.98V$. It is also assumed that the pressure in the tank remains constant. Assuming that the pressure in the tank is 10 pounds, gage, that the temperature is 70 degrees, and that the diameter of the orifice is 1½ inches, $p = 10 + 14.7 = 24.7$, $p_1 = 14.7$, $T = 460 + 70 = 530$; then, $\left(\frac{14.7}{24.7}\right)^{0.2907} = 0.85996$, and $1 - 0.85996 = 0.14004$; whence, $V = 108.67\sqrt{530 \times 0.14004} = 936.2$ feet per second. The discharge is at the rate of $\frac{0.7854 \times 1.5^2}{144} \times 936.2 = 11.48$ cubic feet per second = 688.8 cubic feet per minute. The actual discharge may be taken as $0.98 \times 688.8 = 675$ cubic feet per min.

Discharge of Water through Pipe. About one-half mile from a manufacturing plant there is a large spring; the plan is to tap this spring and lead the water to the plant through a 1½-inch pipe; how many gallons of water per minute will be obtained? The difference of level between the spring and the tank is about 60 feet.

Solution: There are many formulas for calculating the discharge of water through a pipe; some of them are quite complicated, and all are, and must of necessity be, approximate. It is impossible to derive a formula that will fit any case. Pipes or conduits are made of various materials, and the friction of the moving water varies greatly with the material of which the pipe is composed. Even for a particular material, the discharge will not be the same for a pipe that has been in use a long while as for a new pipe. The impurities carried by the water stick to the pipe, causing it to become foul; this reduces the diameter and discharge, and also alters the resistance due to friction. If the slope is not gradual and even, air will accumulate at different points; this also reduces the discharge, since the area of the cross-section at those points is less. Bends, especially those of short radius, reduce the velocity and, consequently, the discharge. Contractions and enlargements, likewise, exert a deterrent effect.

As a result of the examination and comparison of a large number of experiments, the following formula has been derived; it is simple in form, is said to give good results, and is admirably adapted to logarithmic computation:

$$v = 0.0757cd^{\frac{2}{3}}\left(\frac{h}{l}\right)^{\frac{1}{2}}$$

in which v = velocity, in feet per second; d = diameter of pipe, in inches; h = head, in feet; l = length of pipe, in feet; and c = a constant the value of which depends on the material of which the pipe is composed. For new, smooth, wrought-iron pipe, laid straight and without bends, c may be taken as 160. Since the actual internal diameter of a 1½-inch pipe is 1.61 inch, the velocity of discharge in the pipe is

$$v = 0.0757 \times 160 \times 1.61^{\frac{2}{3}} \times \left(\frac{60}{2640}\right)^{\frac{1}{2}} = 2.508 \text{ feet per second}$$

The number of cubic feet per minute discharged is

$$\frac{60 \times 2.508 \times 0.7854 \times 1.61^2}{144} = 2.127$$

$$2.127 \times 7.48 = 16 \text{ gallons per minute}$$

Conductivity of Wrought-iron Pipe. In connection with the design of a cooling apparatus for oil, in which the oil is cooled from a temperature of 100 degrees F. to 60 degrees F. by circulating through a coil placed in a water bath, it is required to find the approximate number of British thermal units (B.T.U.) that will pass through the walls of a 1¼-inch iron pipe, per hour, per square inch of radiating surface, for each degree difference of temperature of the liquid inside and outside of the pipe.

Solution: The coefficient of conductivity of iron, that is, the quantity of heat in therms (252 therms = 1 B.T.U.) that will pass through one square centimeter of surface, one centimeter thick, in one second, per each centigrade degree difference in temperature, is (for temperatures here dealt with) on an average 0.165. Transforming this to English units, the quantity of heat in B.T.U. that will pass through one square inch of surface, one inch thick, in one second, for each degree F. difference in temperature is

$$\frac{0.165 \times 2.54 \times 2.54 \times 5}{252 \times 2.54 \times 9} = 0.00092 \text{ B.T.U.}$$

As 1¼-inch wrought-iron pipe has a thickness of 0.140 inch, the quantity of heat that will pass through its walls per square inch of surface per hour for each degree F. difference in temperature equals:

$$\frac{0.00092 \times 60 \times 60}{0.14} = 23.6 \text{ B.T.U.}$$

Diameters of Suction and Discharge Pipes of Centrifugal Pump. It is required to pump 12 cubic feet of water per minute with a centrifugal pump, raising it 35 feet, 15 feet by suction and 20 by discharge pressure. What will be the diameter of suction and discharge pipes required?

According to a formula by Fink:

$$d = 0.36\sqrt{\frac{Q}{\sqrt{2g\,(h + h_1)}}}$$

in which Q = quantity of water, in cubic feet, pumped per minute;
g = acceleration due to gravity = 32.16;
h = height of suction in feet;
h_1 = height of discharge in feet;
d = diameter of suction and discharge pipe, in feet.

Inserting the known values in the given formula we have:

$$d=0.36\sqrt{\frac{12}{\sqrt{2\times 32.16\,(15+20)}}}=0.36\sqrt{\frac{12}{47.45}}=0.36\times 0.5=0.18,$$

approximately.

A pipe 0.18 foot, or $2\frac{1}{8}$ inches, in diameter would be required.

Bursting Pressure of Pipes. The bursting pressure of pipes can be determined approximately by the following formula (Barlow's):

$$P=\frac{2T\times S}{O}$$

in which P = bursting pressure, in pounds per square inch;
T = thickness of wall, in inches;
O = outside diameter of pipe, in inches;
S = tensile strength of material, in pounds per square inch.

The value of S, as determined by actual bursting tests, is 40,000 pounds for butt-welded steel pipe, and 50,000 pounds for lap-welded steel pipe.

Short Rule for Calculating Weight of Water. An easy, accurate rule for calculating the weight of water when the volume is given in cubic inches follows:

The weight of water varies considerably with the temperature; also, any impurities held in solution will affect its weight. In practice, it is sufficiently accurate to take the weight of a cubic foot as 62.4 pounds. In this case, 1 cubic inch weighs $\frac{62.4}{1728}=\frac{13}{360}$ pound $=\frac{12}{360}+\frac{1}{360}=\frac{1}{30}+\frac{1}{30}\times\frac{1}{12}$ pound. Hence, divide the number of cubic inches by 30; divide the quotient by 12; then add the two quotients. For example, what is the weight of a gallon of water? One gallon contains 231 cubic inches; $231 \div 30 = 7.7$; $7.7 \div 12 = 0.64\frac{1}{6}$; $7.7 + 0.64\frac{1}{6} = 8.34\frac{1}{6}$ pounds. Note that the fraction $\frac{13}{360}$ is convenient for use on the slide-rule.

Upward Pressure of Submerged Tank. A cylindrical tank that is 7 feet in diameter and 30 feet long is sunk in water to within 1 foot of the top; what is the pressure tending to raise the tank? The weight of the tank is 4 tons.

Solution: The pressure tending to raise a body that is submerged or partly submerged is always equal to the weight of the liquid displaced. Assuming that the axis of the tank is horizontal, the volume of water displaced is the same as that of a solid having a base ACB (Fig. 5) and an altitude of 30 feet, the length of the tank. First calculate the area of the segment ADB and subtract

this from the area of the circle $ACBD$; the remainder will be the area of the segment ACB. The radius $OA = 7 \div 2 = 3.5$ feet; the height of the segment is $ED = 1$ foot; hence, $OE = 3.5 - 1 = 2.5$ feet, and $\cos AOD = 2.5 \div 3.5 = 0.7142857$. The angle corresponding to this cosine is 44 deg., 24 min., 55 sec., and $AOB =$ 88 deg., 49 min., 50 sec. = 1.5503857 radian. The area of the segment ADB is $\frac{1}{2}r^2 (V - \sin V) = \frac{1}{2}r^2 (1.5503857 - 0.9997917) =$ 3.3724 square feet, V being the central angle AOB. Area of $ACB = \pi \times 3.5^2 - 3.3724 = 35.1122$ square feet. Taking the weight of a cubic foot of water as 62.5 pounds, the weight of the water displaced is $35.1122 \times 30 \times 62.5 = 65{,}835$ pounds, say 33 tons. As the tank weighs only 4 tons, an additional weight of $33 - 4 = 29$

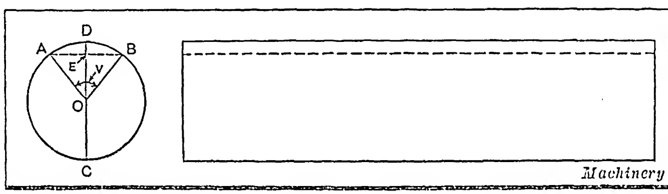

Fig. 5. To find Pressure tending to raise Tank of Given Diameter and Length submerged to Given Depth

tons must be placed on the tank in order to sink it to the required depth.

Capacity of a Disk Fan. How many cubic feet of air does a disk fan, 30 inches in diameter, deliver when running at a speed of 500 revolutions per minute?

For this problem use the following formula:

$$C = 0.6DRA$$

in which C = cubic feet of air delivered per minute;
D = diameter of fan in feet;
R = revolutions per minute;
A = area of fan in square feet.

In the given problem D, in feet $= \frac{30}{12} = 2.5$; $R = 500$; and $A = D^2 \times 0.7854 = 2.5^2 \times 0.7854 = 4.909$. Inserting these values in the formula:

$$C = 0.6 \times 2.5 \times 500 \times 4.909 = 3{,}681.75 \text{ cubic feet}$$

Thickness of a Cylindrical Shell. The formula for finding the thickness of a cylindrical shell that is subjected to internal fluid pressure is derived as explained in the following:

For convenience, assume that the fluid is steam or gas (say compressed air) and that its tension (pressure) is p pounds per square inch. Denote the length of the cylinder by l, the thickness of the shell by t, and the interior diameter by d. The illustration, Fig. 6, represents a cross-section perpendicular to the axis of the cylinder. It will be assumed that this cross-section represents a cylinder having a lenght of 1 inch.

According to Pascal's law, the pressure at any point is always perpendicular to the surface at that point; consequently, it is always radial, as indicated by the arrows AB, HD, and FE, which represent the pressure p on a unit of area at B, D, and E, respectively. If the pressure is great enough, it will separate one half of the shell from the other half; suppose it separates the upper half MNP from the lower half MDP. Since the forces acting downward are equal and opposite to those acting upward, it will suffice to determine the downward pressure. At D the unit pressure acts entirely downward; at B it can be resolved into two components, one AC acting downward, and the other CB acting horizontally. As M and P are approached, the downward pressure becomes less and less, and when M (or P) is reached it becomes 0. By the methods of calculus, it is shown that the total downward pressure on the strip is $p \times d$, and on the shell it is $p \times d \times l$. This pressure is resisted by the strength of the material of the shell multiplied by the area of the ruptured surface, or $2t \times l \times S$, in which $2t \times l$ is the area in square inches (l being the length in inches) and S is the ultimate strength in pounds per square inch. Therefore,

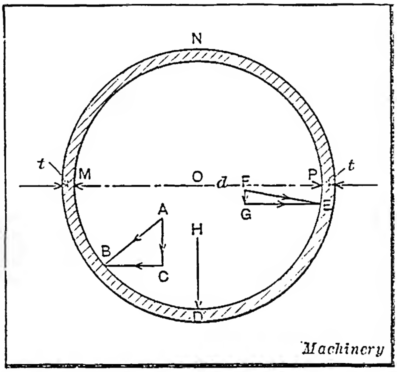

Fig. 6. To find the Thickness t of a Cylindrical Shell subjected to Internal Fluid Pressure

$$2tSl = pdl, \qquad \text{or} \qquad t = \frac{pd}{2S}$$

This formula presupposes that t is small, compared with $r = \frac{d}{2}$, and that l is large, compared with r. The formula may be solved for p, giving

$$p = \frac{2tS}{d} = \frac{tS}{r}$$

If t is greater than $0.1r$, or $\frac{r}{10}$, it is best to use the following formulas:

$$p = \frac{tS}{r+t} \qquad \text{and} \qquad t = \frac{pr}{S-p}$$

It is best to calculate t by the first formula, and then, if it is greater than $0.1r$, recalculate it by the second formula. In practice, S should always be divided by the proper factor of safety before it is substituted in any of the foregoing formulas.

Thickness of Cylinders for High Pressures. Numerous formulas have been derived for the guidance of designers of thick cylinders. The required thickness may be determined by either of the following formulas:

$$T = R\left(\sqrt{\frac{H+P}{H-P}} - 1\right) \tag{1}$$

$$T = R\left(\sqrt{\frac{H+0.4P}{H-1.3P}} - 1\right) \tag{2}$$

where T = thickness of cylinder wall;
H = safe fiber stress of metal;
P = unit working pressure of fluid in pounds per square inch;
R = inner radius of cylinder.

Formula (1) (Lamé), which is generally used in the United States, has been developed in a theoretical way; Formula (2) (Bach), generally accepted in Germany, has been derived by experiment. They give nearly the same results if the value of P is only a small part of H; for instance, when $H = 7000$ pounds per square inch, and $P = 1500$ pounds per square inch. For higher values of P, Formula (2) indicates thicker walls than Formula (1), and as a result limits the use of high pressures sooner, in accordance with practical experience. Both formulas are valid only for internal pressure, and cannot be applied in designing a part such as a ram or plunger which is exposed to external pressure. For this case there is only one formula (Bach), which is as follows:

$$T = \frac{D}{2}\left(1 - \sqrt{\frac{C-1.7P}{C}}\right) \tag{3}$$

where C = safe compressive stress of metal;
D = diameter of ram.

The allowable stresses in parts of hydraulic presses are much higher than for other classes of machines, partly on account of the absence of shock. The following values are only to be used with the preceding formulas. For cast iron, the safe tensile stress may be assumed to be between 2500 and 5000 pounds per square inch; and the safe compressive stress between 10,000 and 15,000 pounds per square inch. The upper limits, of course, assume first-class material and foundry work. For cast steel, the corresponding figures are: 10,000 to 15,000 pounds per square inch for tension, and up to 20,000 pounds per square inch for compression.

Formulas (2) and (3) relate strictly to the cylindrical parts of the castings, but a plunger with a hemispherical bottom has greater strength than a tube, and, therefore, requires less thickness. It is a practical foundry rule, however, to make the bottom at least as thick as the cylindrical part, and often even thicker, making an allowance for the shifting of the core and the introducing of the boring bar through a hole.

Example: The following example illustrates the application of Formula (1): Find the thickness of a cast iron cylinder to withstand a pressure of 1,000 pounds per square inch; the inside diameter of the cylinder is to be 10 inches, and the maximum allowable fiber stress per square inch 4,000 pounds.

Inserting the given values in the formula:

$$T = \frac{10}{2}\left(\sqrt{\frac{4000 + 1000}{4000 - 1000}} - 1\right) = 5\,(\sqrt{1.667} - 1) =$$

$$5\,(1.29 - 1) = 5 \times 0.29 = 1.45, \text{ or say } 1\tfrac{1}{2} \text{ inches.}$$

Load Capacity of Helical Spring. What is the load capacity of a helical spring having an outside diameter of 5 inches, made from ½-inch round steel? The tensile stress per square inch of section of spring must not exceed 80,000 pounds.

The formula for the carrying capacity of helical springs is:

$$P = \frac{Sd^3}{2.55D}$$

in which P = safe carrying capacity;
S = safe tensile stress per square inch;
d = diameter of wire,
D = mean diameter of spring = outside diameter minus diameter of wire.

In the given problem $S = 80{,}000$; $d = \frac{1}{2}$; and $D = 5 - \frac{1}{2} = 4\frac{1}{2}$. If these values are inserted in the formula we have:

$$P = \frac{80{,}000 \times 0.5^3}{2.55 \times 4.5} = \frac{10{,}000}{11.475} = 871 \text{ pounds}$$

Moment of Inertia of a Section. The problem is to determine the moment of inertia about the axis X'–X (Fig. 7) which passes through the center of gravity of the rectangle $ABCD$ and is perpendicular to the long side AD. The moment of inertia of a section about any axis is equal to the moment of inertia about a parallel axis through the center of gravity plus the product of the area of the section by the square of the distance between the axes. Let

I = the required moment of inertia;

I_1 = moment of inertia about a parallel axis through the center of gravity;

A = area of section; and

h = perpendicular distance between the axes.

Then

$$I = I_1 + Ah^2$$

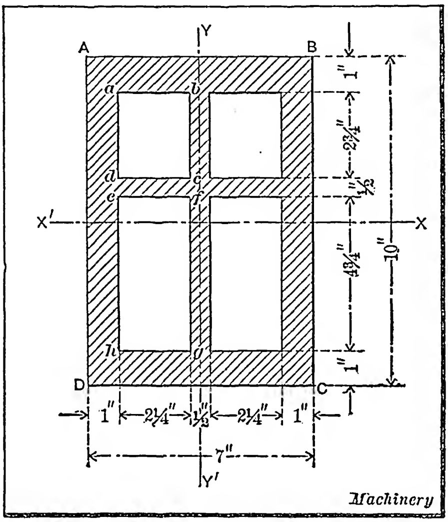

Fig. 7. Diagram illustrating Method of finding Moment of Inertia of a Section

If the given area is divided into sections of such shape that their areas, A_1, A_2, A_3, etc., the distances of their centers of gravity from the given axis, h_1, h_2, h_3, etc., and their moments of inertia, I_1, I_2, I_3, etc., may be calculated, then $I = I_1 + A_1h^2_1 + I_2 + A_2h^2_2 + I_3 + A_3h^2_3 +$ etc.

It is possible to proceed in several ways. Thus, assume that the given section is made up of the following: Two rectangles 1 inch by 10 inches, two rectangles 5 inches by 1 inch, one rectangle ½ inch by 8 inches, and two rectangles 2¼ inches by ½ inch. For the first and third sets, the centers of gravity lie on the axis X'–X, and h_1 and h_3 are both equal to 0; for the second set, $h_2 = 4½$ inches; and for the fourth set, $h_4 = 1$ inch. Since the moment of inertia of a rectangle is $\frac{bd^3}{12}$, when the axis passes through the center

of gravity parallel to the base, and b is the breadth AB and d is the depth AD, $I = \frac{1 \times 10^3}{12} \times 2 + \left[\frac{5 \times 1^3}{12} + 5 \times 1 \times (4\tfrac{1}{2})^2\right] \times 2 + \frac{\frac{1}{2} \times 8^3}{12} + \left[\frac{2\frac{1}{4} \times (\frac{1}{2})^3}{12} + 2\tfrac{1}{4} \times \tfrac{1}{2} \times 1^2\right] \times 2 = 393.63.$

Another method is to calculate the moment of inertia of the rectangle $ABCD$ and then subtract the moments of inertia of twice the rectangle $abcd$ and twice the rectangle $efgh$, all taken with reference to the axis X'–X. Either method will give the same result. If I is calculated for the axis Y'–Y, what was the breadth of the rectangle becomes the depth, and it will be found that I for Y'–Y is much less than it is for X'–X. It is for this reason that a beam is stronger when the long side is vertical.

Length of Rope for Drums. When a rope is wound on a drum the direction of the helix changes with every added layer so that the rope in the second layer will not lie in the groove between the coils of the first layer but must lie across these coils at an angle. This will make the distance between layers nearly equal to the diameter of the rope, so that for practical purposes it is allowable to use the diameter of the rope as the radial pitch of the coils.

The number of coils in a layer will be one less than the width of the drum divided by the diameter of the rope, which will be evident when considering the winding of a rope 1 inch in diameter on a drum 2 inches wide. There is room for but one coil in a layer if the rope forms a true helix. The coils in successive layers will be seen to cross every time at a point opposite the starting point. This will make the increase in radius per layer less than the diameter of the rope at the starting point of a coil, and equal to it at a point opposite the starting point, making each added coil eccentric relative to the first. For such a condition the diameter of the rope must be taken as the radial pitch of the coils, or the most eccentric side will project beyond the flanges. It will also be seen that if the drum is increased from 2 inches to 2½ inches in width, there will be room for 1½ coils under the same conditions. In the case of the 1½-coil condition, the coils will cross at opposite sides in alternate layers, so that a dimension slightly less than the diameter of the rope should be used as the radial pitch of the layers. On a drum having a small number of coils per layer, there will be an error in the true helix, but on a longer drum any variation from the true helix will be very slight. In practice, it is not likely that eccentric windings will cause trouble.

The following formulas contain certain theoretical errors which are believed to be immaterial for most practical purposes. No account is taken of the extra length due to the helix of each coil; of the extra fraction of a coil in each layer due to the wedging action against the flanges; or of the rope lying in the groove of the under layer during a part of each coil. The last error works both ways, as it will in some cases allow an extra layer, but will slightly shorten each coil owing to its shorter radius when in the groove. It is only under very favorable conditions that a large number of layers is wound close enough to crowd more rope on the drum than the following formulas indicate; furthermore, for a few layers the errors are so minute that it is believed these formulas will answer the purpose in all ordinary cases.

In the following formulas,

d = diameter of rope, in inches;
D = diameter of drum, in inches;
F = diameter of drum flange, in inches;
W = width of drum, in inches;
N = number of layers;
O = total length, in feet, of rope in one coil taken from each layer;
Q = number of coils per layer;
L = length of rope, in feet, that can be wound on the drum = OQ

Then

$$N = \frac{F - D}{2d}$$

This formula gives the number of layers to the nearest whole number, unless it is not allowable for any part of the rope to project beyond the flanges of the drum. If such is the case, the fractional remainder should be dropped and the whole number used. If the diameters of the drum and of the flange are considered as the inner and outer circles, repectively, of a ring, the area of this ring may be used to obtain the value of O; thus,

$$O = \frac{0.7854\,(D + 2Nd + D)\,(D + 2Nd - D)}{12d}$$
$$= 0.2618N(D + Nd)$$

$$Q = \frac{W}{d} - 1$$

$$L = OQ = 0.2618N\,(D + Nd) \times \left(\frac{W}{d} - 1\right)$$

Assuming that the diameter of the drum equals 12 inches, the width 6 inches, the diameter of the flange 18½ inches, and the diameter of the rope ⅜ inch, we have:

$$N = \frac{18.5 - 12}{2 \times 0.375} = 8\tfrac{2}{3} \text{ or, say, } 9 \qquad Q = \frac{6}{0.375} - 1 = 15$$

Then,

$$L = 15 \times 0.2618 \times 9\,(12 + 9 \times \tfrac{3}{8}) = 543.4 \text{ feet}$$

Tractive Power of Locomotive and Diameter of Cylinders. Find the tractive power of a simple locomotive having 22-inch cylinder diameters, 26-inch stroke, a boiler pressure of 200 pounds, and 60-inch diameter driving wheels.

The formula for the tractive force of a locomotive is:

$$T = \frac{0.85Pd^2s}{D}$$

in which T = tractive force in pounds;
P = boiler pressure in pounds per square inch;
d = diameter of cylinders in inches;
s = length of stroke in inches;
D = diameter of driving wheels.

Inserting the known values in this formula gives;

$$T = \frac{0.85 \times 200 \times 22^2 \times 26}{60} = 35{,}655 \text{ pounds}$$

Find the diameter of the cylinders of a simple locomotive, having a tractive force of 30,000 pounds; length of stroke, 22 inches; diameter of driving wheels, 57 inches; and boiler pressure, 180 pounds.

The formula for the cylinder diameter is:

$$d = \sqrt{\frac{T \times D}{P \times 0.85 \times s}}$$

in which the letters denote the same quantities as in the preceding formula.

If we insert the known values $T = 30{,}000$; $D = 57$; $P = 180$; and $s = 22$, in the formula, we have:

$$d = \sqrt{\frac{30{,}000 \times 57}{180 \times 0.85 \times 22}} = \sqrt{508.02} = 22.54 \text{ inches,}$$

or, approximately, 22½ inches diameter.

Elevation of Outer Rail on a Railway Curve. When a body moves in a circle, the centrifugal force is expressed by the formula

$$F = \frac{Wv^2}{gR}$$

in which F = centrifugal force, in pounds;
W = weight of moving body, in pounds;
v = velocity of moving body, in feet per second;
g = 32.16;
R = radius of circle or curve, in feet.

Now, when a train goes around a curve, it is deflected from the straight line in which it would ordinarily move and is kept to the rails by reason of the flanges of the outer wheels pressing against the outer rail; this pressure is the centripetal force, and acts in the

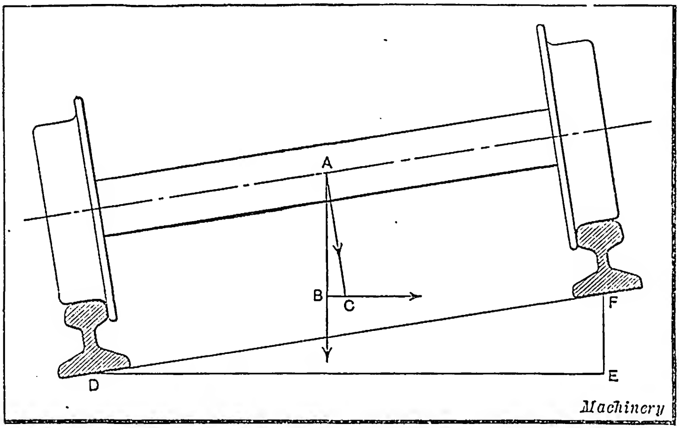

Fig. 8. To find the Elevation of the Outer Rail for a Given Track Curvature and Train Speed

direction BC (see Fig. 8). The weight on a pair of wheels acts vertically downward in the direction AB; the resultant of these two forces is represented by AC. When one of the rails is elevated until the line DF, which is parallel to a line touching the tops of the rails, becomes perpendicular to the line of action of the resultant AC, the centripetal force BC will be exactly balanced by the tendency of the wheels and their load to slide in the direction FD, and there will be no pressure between the flange and the rail. Since the triangles ABC and DEF are similar, $AB : BC = DE : EF$, or $W : F = b : h$, when $b = DE$ and $h = EF$; whence, $h = \frac{Fb}{W}$. Substituting the value of F given previously,

$$h = \frac{Wv^2}{gR} \times \frac{b}{W} = \frac{bv^2}{gR}$$

If b is in inches, h is also in inches. Note that the weight of the train has no effect on the elevation of the rail. Suppose the train is running fifty miles per hour around an eight-degree curve, and that b, or the distance from center to center of rails, is 4 feet, 8.5 inches, or 56.5 inches. The elevation is then

$$h = \frac{56.5\left(\frac{50 \times 5280}{60 \times 60}\right)^2}{32.16 \times 716.2} = 13.2 \text{ inches}$$

The radius of an eight-degree curve is 716.2 feet.

Sign of a Bending Moment. The sign of the bending moment for a simple beam is always positive, or +, and for a cantilever it

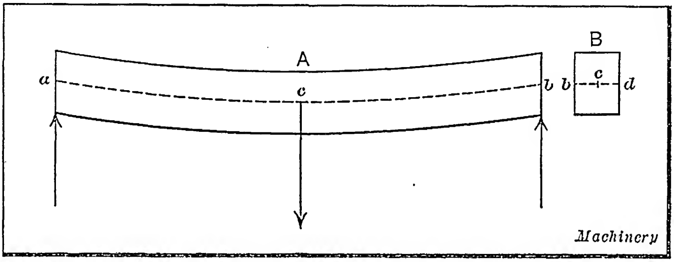

Fig. 9. Diagram showing Condition of Simple Beam subjected to Bending

is negative, or −; the reason for this practice and the meaning of the negative sign will be explained.

Fig. 9 illustrates a heavy beam of rectangular cross-section, supported at both ends as shown, thus representing to all intents and purposes a simple beam uniformly loaded. Let c be the center of gravity of the beam; then all the weight may be considered as concentrated at this point, and the effect produced by this weight is a bending of the beam as indicated in the illustration (greatly exaggerated). The dotted line (ab in A and bd in B) represents the neutral surface; and when the beam lies on a horizontal plane, this neutral surface is horizontal; the neutral surface always passes through the center of gravity of the cross-section. An examination of Fig. 9 shows that all longitudinal fibers of the beam above the neutral surface are in compression, due to bending, and all below the neutral surface are in tension, while those in the neutral surface

are neither shortened nor lengthened — their lengths remain unchanged.

In Fig. 10, which represents a beam fixed at one end only, i.e., a cantilever, an exactly opposite effect is produced; here the fibers above the neutral surface are in tension, and those below are in compression. Since the bending moments are measures of the effects just described, it is customary to call the bending moment of a simple beam positive, and of a cantilever, negative. The negative sign has no significance in so far as it affects calculations pertaining to the strength of beams; but it is useful and necessary in connection with investigations relating to restrained and continuous beams. It will also be noticed that the upper surface of

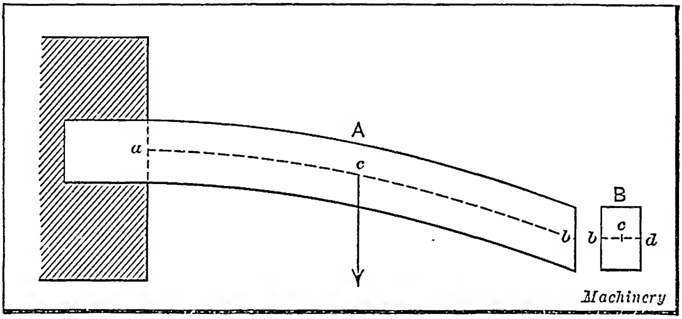

Fig. 10. Diagram illustrating Cantilever Beam subjected to Bending

the beam in Fig. 9 is concave, while in Fig. 10 it is convex; this is another reason for the opposite signs of the bending moments.

Maximum Bending Moment of a Beam. The diagram, Fig. 11, shows a simple beam AB, 25 feet long, which carries a uniform load (including its own weight) of 32 pounds per foot and several concentrated loads distributed as shown. Find the maximum bending moment.

Solution: It is necessary first to find the reactions of the supports. Denote the reaction of the left support by R_1 and that of the right support by R_2, and take the left support as the center of moments. Noting that the entire weight of the uniform load is $25 \times 32 = 800$ pounds and that the center of gravity is midway between the supports, or 12.5 feet from each, we have: $25R_2 = 800 \times 12.5 + 250 \times 2 + 150 \times 5 + 400 \times 9 + 200 \times 16 + 350 \times 22 = 25{,}750$; whence

$R_2 = 1030$ pounds. Taking R_2 as the center of moments, $25R_1 = 800 \times 12.5 + 350 \times 3 + 200 \times 9 + 400 \times 16 + 150 \times 20 + 250 \times 23 = 28{,}000$; whence $R_1 = 1120$ pounds.

Next, add successively, and in order, the various uniform and concentrated loads until the sum equals or exceeds the left reaction $R_1 = 1120$ pounds, as shown. Note that there are three concentrated loads between the left support and that point on the beam where the sum of the loads on the left of the section at that point is equal to $R_1 = 1120$. The sum of these three loads is 800 pounds; subtract this sum from R_1, obtaining $1120 - 800 = 320$. Now divide this result by the weight of the uniform load per foot, obtaining $320 \div 32 = 10$ feet = the distance from the left support to the dangerous section, marked x in the illustration. The maximum bending moment is then equal to (taking n as the center of moments) the moment of the left reaction minus the moments of all loads on the left, or $M = 1120 \times 10 - 32 \times 10 \times \frac{10}{2} - 250 \times 8 - 150 \times 5 - 400 \times 1 = 6450$ pound-feet $= 77{,}400$ pound-inches.

$$
\begin{array}{lr}
32 \times 2 = & 64 \\
 & 250 \\ \hline
 & 314 \\
32 \times 3 = & 96 \\ \hline
 & 410 \\
 & 150 \\ \hline
 & 560 \\
32 \times 4 = & 128 \\ \hline
 & 688 \\
 & 400 \\ \hline
 & 1088 \\
32 \times 7 = & 224 \\ \hline
 & 1312
\end{array}
$$

That this method is correct is easily shown. Thus, draw the vertical line am through the point of left support, and on this line (to any convenient scale) lay off $ar = R_1$ and $rm = R_2$; the line am is called the load line. On this line, lay off a–$1 = 32 \times 2$ (the first section of the uniform load), 1–2 = 250 (the first concentrated load), 2–3 = $32 \times 3 = 96$ (the second section of the uniform load), 3–4 = 150 (the second concentrated load), and so proceed until all the loads have been laid off. Through the points where the various loads are concentrated on the beam, draw vertical lines, as shown, and through the points, 1, 2, 3, etc., on the load line, draw horizontal lines intersecting the vertical lines in the points b, c, d, etc. Connect these points by the broken line *abcdefgthijkl*; this line is called the shear line. Through the point r, draw the horizontal line rs; this line is called the shear axis. In works treating of the strength of materials, it is shown that the dangerous section occurs where the shear line crosses the shear axis; in this case it would pass through the point t in the diagram and the point n on the beam, which lies on the vertical through t. From the method of construction, it is evident that ab, cd, ef, etc., are parallel. Produce hg

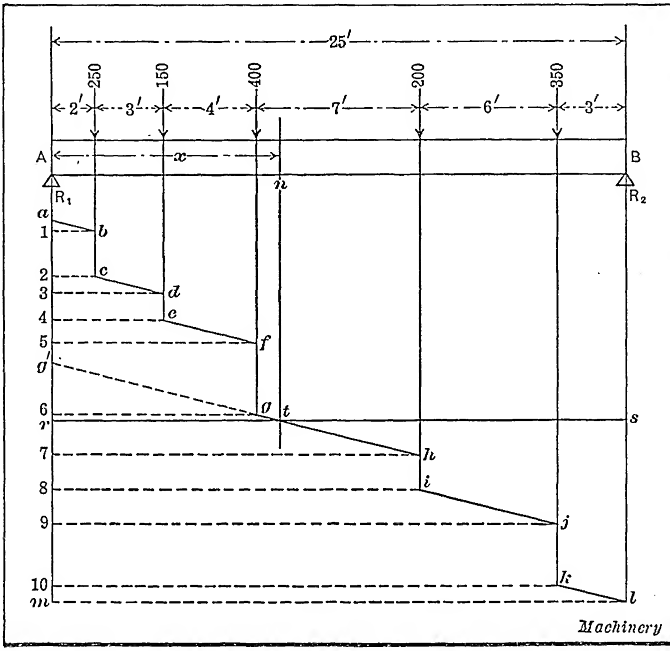

Fig. 11. Diagram illustrating Method of determining Maximum Bending Moment of Beam

until it intersects the load line in g'; then the triangles $a1b$ and $g'r$ are similar right triangles, and

$$\frac{b1}{1a} = \frac{tr}{rg'} \qquad \text{or} \qquad tr = x = \frac{b1 \times rg'}{1a}$$

But, $b1 = 2$, and $1a = 64$; hence,

$$\frac{b1}{1a} = \frac{2}{64} = \frac{1}{32}$$

the reciprocal of the uniform load per foot. Finally, rg' is evidently the uniform load from A to the dangerous section at n.

Dimensions of Foundation Bolts. A 3-inch foundation bolt is to have an enlarged lower end to receive a steel cotter-pin. (See Fig. 12.) It is required to compute the necessary dimensions so

that the lower end of the bolt and the cotter-pin will have the same strength as the smaller section of the bolt in which the thread is cut. The bolt is made of wrought iron which has a tensile strength of 50,000 pounds per square inch, a shearing strength of 45,000 pounds per square inch and a crushing strength of 50,000 pounds per square inch. The cotter-pin is made of steel which has a shearing strength of 68,000 pounds per square inch and a crushing strength of 75,000 pounds per square inch.

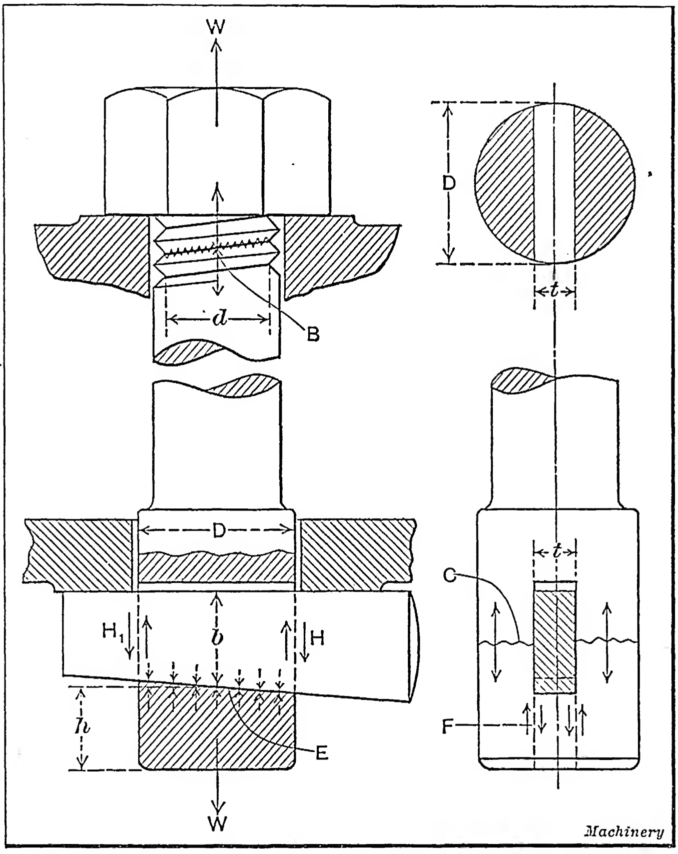

Fig. 12. Foundation Bolt and Different Ways that Failure may occur due to Excessive Stresses

Solution: Taking the working stresses as given, we have:

S_t = tensile strength of wrought iron = 50,000 pounds per square inch;

S_c = crushing strength of wrought iron = 50,000 pounds per square inch;

S_s = shearing strength of wrought iron = 45,000 pounds per square inch;

S_c' = crushing strength of steel = 75,000 pounds per square inch;

S_s' = shearing strength of steel = 68,000 pounds per square inch.

Let:

d = diameter of bolt at root of thread = 2.629 inches;

a = area of cross-section of bolt at root of thread = 5.429 square inches;

D = diameter of enlarged section of bolt;

A = area of enlarged section of bolt;

t = thickness of key;

b = width of key at middle of keyway;

h = depth of bolt below key at middle of keyway;

W = total axial load which bolt can sustain = $S_t \times a$ = 271,450 pounds.

For maximum economy of metal, all sections of both the bolt and key which are under strain should be equally strong, or should have the same strength as that of the bolt at the root of the thread. Failure may occur by:

1. Rupture of bolt at root of thread, as shown at *B*, Fig. 12. The total stress resisting rupture is equal to the product of the area *a* of the bolt and the tensile strength:

$$W = S_t \times a = S_t \times 0.7854d^2 = 39{,}270d^2 \qquad (1)$$

2. By rupture of the slotted section of the bolt, as shown at *C*. The resisting stress is equal to the product of the area of the cross-section of the bolt and the tensile strength:

$$W = S_t(A - Dt) = S_t(0.7854D^2 - Dt) = 39{,}270D^2 - 50{,}000Dt \qquad (2)$$

3. By the crushing of the metal at the bottom of the keyway, as shown at *E*. The resistance to this crushing is equal to the area of the base of the keyway times the crushing strength:

$$W = S_c \times D \times t = 50{,}000Dt \qquad (3)$$

4. By the shearing of a vertical section of the bolt below the keyway and equal in width to the key, as shown at *F*. Since both sides of this section must be sheared simultaneously, the resistance

to shearing is equal to the product of the shearing strength and twice the area of one side:

$$W = S_s \times 2\,(D \times h) = 90{,}000Dh \tag{4}$$

5. By crushing the key at the contact-surface between it and the keyway, as shown at E. The total resisting stress is equal to the product of the crushing strength of the metal of the key and the area of the keyway, or

$$W = S_c'\,(D \times t) = 75{,}000Dt \tag{5}$$

The same action occurs at the bearing of the key on the foundation plate. The area of bearing surface at these points should be at least equal to that of the keyway.

6. By shearing the key where it enters and leaves the bolt, as shown at H and H_1. Since there are two sections of the key to be sheared simultaneously, the total stress resisting shearing is equal to the product of the shearing strength of the metal by twice the mean cross-sectional area of the key:

$$W = S_s' \times 2\,(b \times t) = 136{,}000bt \tag{6}$$

To obtain a strength at each of these various sections under strain, which shall be equal to that of the bolt at the root of the thread, it is evident that the equation giving the resisting stress in each case must be equated with Equation (1), and then the unknown values D, t, b, and h can be found by successive substitutions. Thus, to find the diameter D, equating Equations (5) and (1) we have:

$$75{,}000Dt = 39{,}270d^2$$
$$Dt = 0.524d^2 \tag{7}$$

Equating Equations (2) and (1), we have:

$$39{,}270D^2 - 50{,}000Dt = 39{,}270d^2$$

Substituting and transposing in the preceding expression and in Equation (7) we have:

$$39{,}270\,D^2 = 65{,}470d^2$$
$$D^2 = 1.66d^2$$
$$D = 1.29d = 3.39 \text{ inches or, say, } 3\tfrac{7}{16} \text{ inches} \tag{8}$$

To find the thickness t, we proceed by equating Equations (5) and (1):

$$75{,}000Dt = 39{,}270d^2$$

Since $D = 1.29d$, we have by substituting and transposing:

$$96{,}750dt = 39{,}270d^2$$

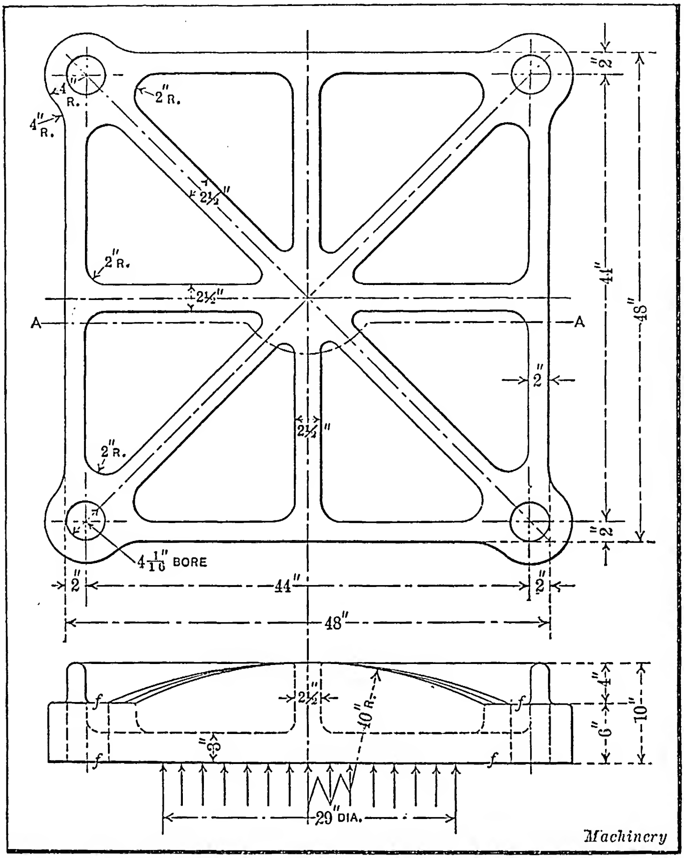

Fig. 13. Dimensions of Ribbed Plate, the Safe Load of which is required

$$t = 0.406d = 0.406 \times 2.629 = 1.067 \text{ inch or, say, } 1\tfrac{1}{8} \text{ inches} \quad (9)$$

To find the width b, we proceed by equating Equations (6) and (1):

$$136{,}000bt = 39{,}270d^2$$

Since $t = 0.406d$, we have by substituting and dividing by d:

$$55{,}216b = 39{,}270d$$

$$b = 0.711d = 0.711 \times 2.629 = 1.869 \text{ inch or about } 1\tfrac{7}{8} \text{ inches} \quad (10)$$

To find the depth h, we equate Equations (4) and (1):

$$90{,}000Dh = 39{,}270d^2$$

Since $D = 1.29d$, we have by substituting and dividing by d;

$$116{,}100h = 39{,}270d$$

$$h = 0.338d = 0.889 \text{ inch or, say, } {}^{15}\!/_{16} \text{ inch.}$$

It will be observed that the values of the various dimensions deduced as above depend wholly on the working stresses assumed. Any change in these stresses will make a corresponding change in

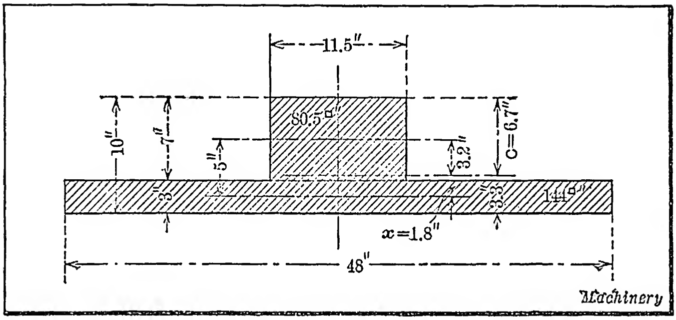

Fig. 14. Equivalent Section through Casting on Line *A-A*

these dimensions. In any event, the latter are subject to some alteration, owing to the conditions of practice.

Strength of Ribbed Plates. The ribbed cast-iron plate shown in Fig. 13 is uniformly loaded over a surface 29 inches in diameter at the center and firmly supported at the four corners. What load will it safely support and at what load will it fail?

There is always more or less uncertainty as to the strength of ribbed cast-iron sections on account of shrinkage strains, blow-holes, and other inherent defects which may develop under load but which may not be apparent to the inspector. This is especially true of those cases where the ribs are under tension, as in the present example. In view of such considerations some designers virtually neglect the ribs when estimating the strength, treating the casting as a flat plate slightly thicker than the actual plate, thus allowing something for the stiffening effect of the ribs. While this rough and ready method has simplicity in its favor, and may be quite satisfactory when applied with discretion, it is not very satisfying from

a technical viewpoint. If the ribs are of no calculable value, why not omit them from the casting as well as from the calculations? The logical method seems to be to calculate the strength based on the full value of the ribs and allow a stress sufficiently low to take into account possible imperfections in both theory and practice.

Assuming the casting to break along the line A–A, the section would be about the equivalent of that shown in Fig. 14. The area

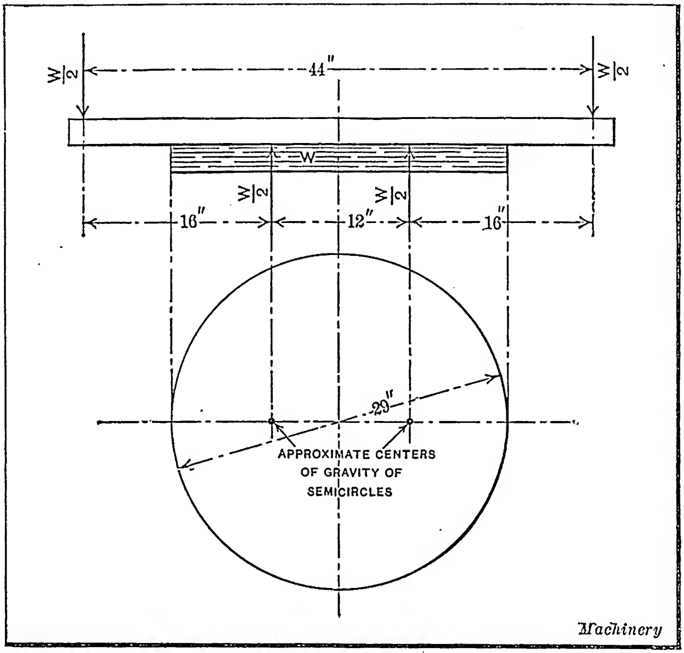

Fig. 15. Diagram illustrating Plate loaded as a Beam

of the section is $A = 7 \times 11.5 + 3 \times 48 = 224.5$ square inches. Taking moments of the areas about the center of the three-inch plate, the distance to the center of gravity is

$$x = \frac{7 \times 11.5 \times 5}{224.5} = 1.8 \text{ inch, approximately}$$

Taking the areas times the square of their distance from the center of gravity, plus their moment of inertia about their own axis, the moment of inertia of the section is

$$I = 80.5 \times 3.2^2 + 144 \times 1.8^2 + \frac{11.5 \times 7^3}{12} + \frac{48 \times 3^3}{12} = 1727$$

The section modulus for the tension side is then

$$Z = \frac{I}{C} = \frac{1727}{6.7} = 258$$

The casting may now be considered as a beam loaded as indicated in Fig. 15, when the bending moment is

$$M = \frac{W}{2} \times 22 - \frac{W}{2} \times 6 = 8W$$

The general formula for the relations of moment M, stress S, and section modulus Z is $M = SZ$. Assuming a value of $S = 4000$ and substituting the values of M and Z gives $8W = 4000 \times 258$ or $W = 129{,}000$ pounds as the safe load based upon the comparatively high working stress in the tension side of the ribs of 4000 pounds per square inch. If the load is suddenly applied or the casting is subjected to shock, about half this value will represent the safe load.

Since the elasticity of cast iron in tension and compression is not the same, $\frac{I}{C}$ does not represent the true section modulus, being only approximately true. Also the expression $M = SZ$ is only supposed to hold good within the elastic limit, and with strict propriety cannot be applied for the breaking load. However, in view of the other uncertainties of the problem, precision is far from attainable and this relation may be assumed to be approximately true for the breaking load also. Taking a stress of 16,000 pounds per square inch, which is four times the assumed working stress, failure would occur at a load of approximately $4 \times 129{,}000 = 516{,}000$ pounds; but if the plate is subjected to shock, a load less than half this amount might cause failure.

Stresses in Wire Rope Due to Bending. According to a general rule used when designing hoisting apparatus, the diameter of the sheave should be not less than 100 times the diameter of a rope having 6 strands, 7 wires to the strand, nor less than 60 times the diameter of a rope having 6 strands, 19 wires to the strand. The actual value of these numbers is somewhat arbitrary, but their ratio should be about as mentioned, that is, 100 : 60. The following will make this clear:

The strength of a wire rope is equal to the strength of one of the wires composing it multiplied by the number of wires in the rope.

When a wire is bent around a cylinder, as shown in Fig. 16, the upper half is lengthened and the lower half is compressed, while the length of the center line is unaltered. Let R = radius of sheave, r = radius of wire, and D and d = corresponding diameters. The topmost element of the wire will then be lengthened by the bending an amount equal to $\pi(R + d) - \pi(R + r) = \pi r$, when the two parts of the rope are parallel, as shown in the illustration. The length of the wire that forms the semicircle is $\pi(R + r)$; hence, the unit elongation, or unit strain, is $s = \frac{\pi r}{\pi(R + r)} = \frac{r}{R + r} = \frac{r}{R} = \frac{d}{D}$, when the r in the denominator is dropped as having no appreciable effect on the ratio.

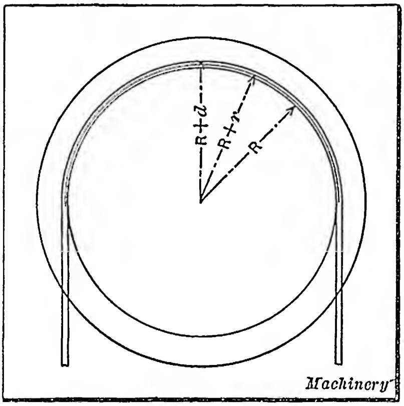

Fig. 16. Diagram for Wire Rope Stress Calculations

Let P = force required to bend wire, A = area of cross-section of wire, and S = unit stress; then, $S = \frac{P}{A}$. The coefficient of elasticity E is defined as the ratio of the unit stress to the unit strain, or $E = \frac{S}{s}$ from which,

$$S = Es = E\frac{d}{D}$$

This expression for the stress due to bending applies only to the topmost element of the wire; and as the various elements are located nearer the center, the stress in each becomes less. A prominent wire-rope manufacturer recommends the formula

$$S = \frac{Ed}{D} \times 0.45$$

which is probably sufficiently close to actual conditions for practical purposes. Since the value of E is always practically the same for the same material, a change in S can be produced only by a change in the ratio $\frac{d}{D}$, and since this ratio is independent of the length of the arc of contact, it follows that the same bending stress will be produced when the rope is bent around a small arc as when

it is bent around an entire circle. It is also evident that if D remains the same, the smaller d is the smaller will be the bending stress and, vice versa, the larger c is the greater the bending stress S.

A 6 by 7 rope contains $6 \times 7 = 42$ wires, and a 6 by 19 rope contains $6 \times 19 = 114$ wires; therefore, for ropes of the same diameter, the wire in a 6 by 7 rope will be larger than in a 6 by 19 rope.

According to the section on wire rope in MACHINERY'S HANDBOOK, $d = 0.106 \times d_1$ for a 6 by 7 rope; and $d = 0.063 \times d_1$ for a 6 by 19 rope, d being the diameter of the component wire and d_1 the diameter of the rope. The ratio of the right-hand members of these two equations is

$$\frac{0.106 \times d_1}{0.063 \times d_1} = \frac{0.106}{0.063} = \frac{1}{0.594} = \frac{100}{59.4}$$

which is almost exactly the same as the ratio of 100 : 60. Some manufacturers permit the ratio $\frac{D}{d}$ to be as low as 80 for a 6 by 7 rope and as low as 48 for a 6 by 19 rope, but the minimum values of 100 and 60 tend to greater safety. For a 1-inch 6 by 7 steel rope, $d = 0.106 \times 1 = 0.106$ inch, and the stress due to bending is $S = \frac{30{,}000{,}000 \times 0.106}{100} \times 0.45 = 14{,}310$ pounds per square inch, since for steel $E = 30{,}000{,}000$ and $D = 100 \times 1 = 100$. The area of the cross-section of the wires of a 1-inch rope is 0.3706 square inch; hence, the total stress P due to bending is $14{,}310 \times 0.3706 = 5303$ pounds $= 2.65$ tons. For a 1-inch 6 by 19 steel rope, $D = 60$ inches and $d = 0.063$ inch; hence, $S = \frac{30{,}000{,}000 \times 0.063}{60} \times 0.45 = 14{,}175$ pounds per square inch; the area is 0.3554; whence, $P = 14{,}175 \times 0.3554 = 5038$ pounds $= 2.52$ tons. These values are a little over one-third of the safe working load of extra-strong, crucible-steel, wire rope.

Centrifugal Stresses. A convenient system for the computation of stresses in rotating bodies due to centrifugal force will be given but without derivation or proof of the various formulas. The fundamental quantity will be c, the centrifugal force in pounds, exerted by a concentrated weight of one pound, whose center of gravity rotates in a circle one inch in diameter, with a speed of N revolutions per minute. We have

$$c = 0.00001421N^2 \qquad (1)$$

One who has much computation work to do will find it convenient to prepare a table of the values of c for the values of N most

frequently used. The centrifugal force in pounds (that is, the inward radial force which must be applied at the center of gravity) for a concentrated mass whose weight is W pounds, and whose center of gravity rotates in a circle d inches in diameter, is expressed by the formula:

$$\text{Centrifugal force} = cWd \tag{2}$$

In computing stresses due to centrifugal force, the fundamental method will be to compute directly F, the "bursting force," in pounds, tending to cause rupture. The stress is the bursting force divided by the area. The bursting force is understood to be the force in pounds applied to each of the two cross-sections of the ring, disk, etc., which are on opposite sides of the axis. That is to say, the total force tending to cause rupture across the whole area on both sides of the axis, cut out by a plane containing the axis, is twice the bursting force.

Rule 1: In order to obtain the stress, the sum of the bursting forces due to all causes must be divided by the area in square inches of the solid metal (deducting holes, etc.) *on one side of the axis only.*

The use of this rule will give only the "average stress" throughout the section. If the cross-section has considerable length in the directions of the radius, the stresses may not be distributed uniformly. Then in various parts of the section, the stress is above and below the average.

Rule 2: For most purposes it is best to use only the "average stress" as given by Rule (1), making the stress and factor of safety such as to allow for any variation of stress above the average which may occur, according to the judgment of the designer.

The section of the disk, ring, etc., should be made such as to give a uniformly distributed stress, whenever possible. For instance, in the case of a disk the thickness (in the direction of the axis) should be much greater at the hub than at the rim. In other words, it is best to place additional metal for resisting stress near the hub where the bursting force of the added metal will be least. The shape of section which will give uniform distribution of stress can be computed mathematically for any case.

Rule 3: In many cases, it will be sufficient if the designer simply thickens the section toward the hub according to his judgment as to what will give uniform stress as nearly as possible under the given circumstances. This is not a scientifically correct procedure, but the mathematical difficulties of the exact methods and the usual

large factor of safety justify it. In this case the factor of safety becomes partly a "factor of ignorance."

The bursting force F due to n concentrated masses (such as arms, gear teeth or poles on a revolving field), each of weight W pounds, whose centers of gravity are uniformly distributed completely around a circle of diameter d inches is expressed by the formula

$$F = \frac{cWd}{2 \sin \frac{180}{n}} \tag{3}$$

This formula gives exactly the bursting force across a section midway between two masses, where it is a maximum. The bursting force across the section at the radius through the center of gravity of a mass is less and is found by using the tangent instead of the sine in Formula (3). The two values are not appreciably different unless n is small. An approximate formula equivalent to (3) and giving nearly correct results where n is greater than 8, and almost exact results for large values of n, is as follows:

$$F = \frac{cWdn}{2\pi} \tag{4}$$

Formulas for the Bursting Force of a Ring, or Disk. Next, let us consider the bursting force due to the metal composing a ring, disk, etc., of any cross-section. As already remarked, the cross-section is the plane figure, on one side of the axis only, cut out by a plane containing the axis.

w = density of metal, pounds per cubic inch;
$W = \pi wAd$ = weight in pounds of the complete disk or ring (whole circumference);
d = diameter in inches of circle through center of gravity of cross-section;
$r = \frac{d}{2}$ = distance in inches from axis of rotation to center of gravity of cross-section;
A = area of cross-section in square inches;
I = moment of inertia of plane figure forming cross-section, about an axis through its center of gravity, parallel to axis of rotation,
$I' = Ar^2 + I$ = moment of inertia of cross-section about axis of rotation;
d_1 = inner diameter of ring or disk in inches;
d_2 = outer diameter of ring or disk in inches;

t = uniform thickness of a flat disk or ring, in the direction of the axis;

F = bursting force in pounds due to metal of ring, disk, etc., applied to cross-section on one side of axis.

For a ring, disk, etc., of any cross-section:

$$F = 2cw\,(Ar^2 + I) \tag{5}$$

or

$$F = 2cwI' \tag{6}$$

or

$$F = \frac{cWd}{2\pi}\left(1 + \frac{4I}{Ad^2}\right) \tag{7}$$

For a ring, whose outer and inner radii are nearly the same, approximate formulas are

$$F = 2cwAr^2 \tag{8}$$

or

$$F = \frac{cwA\,(d_1 + d_2)^2}{8} \tag{9}$$

or

$$F = \frac{cWd}{2\pi} \tag{10}$$

or

$$F = \frac{cW\,(d_1 + d_2)}{4\pi} \tag{11}$$

The stress in a ring whose radii are nearly the same is approximately,

$$\frac{0.3732wV^2}{e} \tag{12}$$

where V is the velocity in feet per second and e is the ratio of the area effective for resisting stress to the average area, that is, the "efficiency of joint."

For a flat ring or disk or cylinder with a hole in it, of uniform thickness in the direction of the axis

$$F = \frac{cwt}{12}\left(d_2^3 - d_1^3\right) \tag{13}$$

or

$$F = \frac{cW}{3\pi}\left(d_1 + d_2 - \frac{d_1 d_2}{d_1 + d_2}\right) \tag{14}$$

Computation of Stress in an Impeller. As an example of the methods of using the preceding formulas, we will compute the stress

in the cast-iron impeller of a centrifugal pump, 34 inches in diameter, rotating at 1,000 revolutions per minute. The impeller consists of a central disk, with bosses at the center forming the hub. On both sides of the disk are cast ribs forming the vanes. The disk is thickened at the hub so as to make the stress uniform according to Rule (3). The thickness in the direction of the axis just outside the hubs is 3 inches. The thickness gradually decreases and becomes ½ inch at the circumference. There are 20 vanes on each side extending from inlet to circumference, and 20 more on each side beginning about half way out and extending to the circumference.

The weight of one of the long vanes was computed to be 4 pounds, and the weight of one of the short vanes was 2 pounds. The diameters to the centers of gravity of the vanes were computed to be 26 and 20 inches respectively. The area of the plane figure forming the cross-section of the disk was found to be $A = 28$ square inches, and the moment of inertia about the axis of rotation $I' = 1143.8$. This data and the preceding formula enable us to compute the stress, as follows:

In the first place, by formula (1)

$$c = 0.00001421 \times 1000 \times 1000 = 14.21$$

The bursting force due to the long vanes on both sides by Formula (4) is:

$$F_1 = \frac{2 \times 14.21 \times 4 \times 26 \times 20}{2 \times 3.1416} = 9408 \text{ pounds}$$

The bursting force due to the short vanes on both sides, by the same formula, is:

$$F_2 = \frac{2 \times 14.21 \times 2 \times 20 \times 20}{2 \times 3.1416} = 3618 \text{ pounds}$$

Next we will compute the bursting force due to the metal of the disk itself, by Formula (6). Assuming that w, the weight of cast iron per cubic inch is 0.26, we have,

$$F_3 = 2 \times 14.21 \times 0.26 \times 1143.8 = 8451 \text{ pounds}$$

The total bursting force is

$$F_1 + F_2 + F_3 = 9408 + 3618 + 8451 = 21{,}477 \text{ pounds}$$

This is resisted by the metal forming the cross-section of the disk, 28 square inches. Hence the stress is 21,477 ÷ 28 or 767 pounds per square inch. This is a proper stress for cast iron under the circumstances, and hence the impeller is safe.

Computation of Stress in a Flywheel. Consider as a second example a six-arm cast-iron flywheel, rotating at 100 revolutions per minute, built in halves and held together by the usual steel link-bars at the rim and by bolts at the hub. Let the outside diameter be 12 feet, the inside diameter of the rim 10 feet, and the thickness of the rim in the direction of the axis 12 inches. The total area on each side of the link bars is 16 square inches, and of the cast iron at the minimum section, 128 square inches. Let the hubs be 12-inch bore and 24 inches long, and of a shape equivalent to an outside diameter of 30 inches. Let the average cross section of the arms be 50 square inches. The fillets at the ends, etc., are such that the weight of each arm is the same as that of an arm with a cross-section of 50 square inches, which is 52 inches long. The estimated radius to the center of gravity of the arms is 35 inches. The hub is held together by two $2\frac{1}{4}$-inch bolts on each side; area at root of thread, 3.023 square inches each.

In the first place, by formula (1)

$$c = 0.00001421 \times 100 \times 100 = 0.1421$$

The weight of an arm is $50 \times 52 \times 0.26 = 676$ pounds. The bursting force due to the arms by Formula (3), is

$$F_1 = \frac{0.1421 \times 676 \times 70}{2 \times \sin 30^\circ} = 0.1421 \times 676 \times 70 = 6724 \text{ pounds}$$

The approximate formula (4) would have given

$$\frac{0.1421 \times 676 \times 70 \times 6}{2 \times 3.1416} = 6421 \text{ pounds}$$

which is considerably in error.

The bursting force due to the hub by Formula (13) is

$$F_2 = \frac{0.1421 \times 0.26 \times 24}{12}(30^3 - 12^3) = 0.1421 \times 0.26 \times 2 \times 25{,}272$$
$$= 1868 \text{ pounds}$$

The bursting force due to the rim, by Formula (13) is

$$F_3 = \frac{0.1421 \times 0.26 \times 12}{12}(144^3 - 120^3) = 0.1421 \times 0.26 \times 1{,}258{,}000$$
$$= 46{,}480 \text{ pounds}$$

The approximate formula (9) would have given $\frac{1}{8} \times 0.1421 \times 0.26 \times 144 \times 264^2 = 46{,}350$ pounds, which is nearly correct.

Let us suppose that the rim takes care of its own bursting force, and the hub supports itself and the arms. The stress on the link bars is then $46{,}480 \div 16 = 2{,}900$ pounds per square inch, and on

the cast iron is 46,480 ÷ 128 = 363 pounds per square inch. The total bursting force on the hub bolts is 6724 + 1868 = 8592 pounds, and the stress is 8592 ÷ 6.046 = 1420 pounds per square inch. These stresses are extremely low and the wheel is therefore of ample strength.

Power-transmitting Capacity of Spur Gearing. What horsepower may safely be transmitted by a cast-iron machine-cut spur gear which has a pitch diameter of 16 inches, a face width of 3 inches, 64 teeth, and a pressure angle of 14½ degrees, assuming that the gear is to run at a velocity of 120 revolutions per minute?

The formulas for the solution of this problem are as follows:

$$V = 0.262DR$$

$$S = S_s \times \frac{600}{600 + V}$$

$$W = \frac{SFY}{P}$$

$$\text{H.P.} = \frac{WV}{33{,}000}$$

in which V = velocity in feet per minute at pitch diameter;
D = pitch diameter in inches;
R = revolutions per minute;
S = allowable unit stress of material at given velocity;
S_s = allowable static unit stress of material;
W = maximum safe tangential load, in pounds, at pitch diameter;
Y = factor dependent upon pitch and form of tooth;
F = width of face of gear;
P = diametral pitch;
H.P. = horsepower transmitted.

The known values to be inserted in the given formulas are $D = 16$, $R = 120$, S_s (for cast iron, assumed) = 6000, $F = 3$, Y (for 64 teeth, standard form) = 0.36, and $P = 64 \div 16 = 4$. If we insert these values, as required, in the formulas given, we have:

$$V = 0.262 \times 16 \times 120 = 503 \text{ feet}$$

$$S = 6000 \times \frac{600}{600 + 503} = 3264 \text{ pounds per square inch}$$

$$W = \frac{3264 \times 3 \times 0.36}{4} = 881 \text{ pounds}$$

$$\text{H.P.} = \frac{881 \times 503}{33{,}000} = 13.4 \text{ horsepower}$$

The value of factor Y depends upon the number of teeth and pressure angle. These factors may be obtained from MACHINERY'S HANDBOOK (see the table, "Factors for Calculating Strength of Gear Teeth").

Design of Back-gearing for Given Ratios. The following method will be found useful for determining the number of teeth in each of the gears of a back-geared train, or in any other train of gears where the center distance is the same for each pair of wheels. The method is based on the assumption that the pitch of the teeth is the same

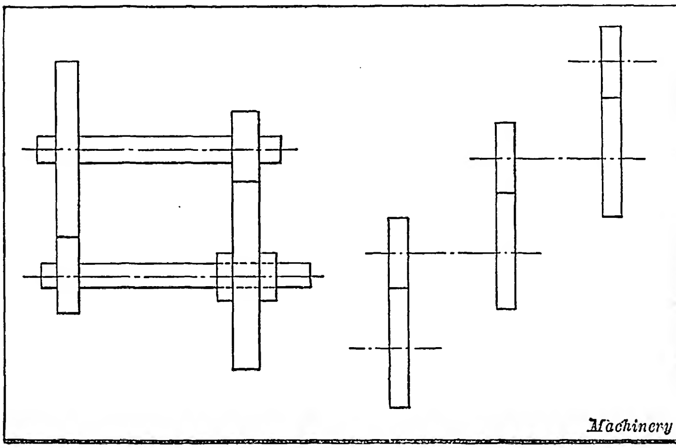

Fig. 17. Back-geared and Triple-geared Drives representing Problems in Design of Gearing for Given Ratios

for all pairs of gears. Suppose it is desired to determine the number of teeth for each of the gears in a set of back-gears which are to have a ratio of $\frac{1}{N}$. The familiar arrangement of a set of back-gears is shown diagrammatically at the left in Fig. 17, and to determine the number of teeth for the wheels, the required ratio $\frac{1}{N}$ is factored into two equivalent ratios as follows:

$$\frac{1}{N} = \frac{1}{O} \times \frac{1}{P}$$

The sum of the number of teeth in either pair of wheels is given by the following formula:

$$\text{Number of teeth} = K\,(O + 1) \times (P + 1)$$

The value of the factor K is such that for the smallest sizes of gears which can be used, it will reduce the product $(O + 1) \times (P + 1)$ to a whole number. There will, of course, frequently be a minimum limit to the size of the smallest gear which will call for the use of a higher value of K than that which is actually required to reduce the preceding product to an integral number.

The use of this method will be better understood by illustrating its application in an actual problem of gear design.

Example 1: Suppose it is required to design a set of back-gears which have a ratio of $\frac{1}{56}$. Factoring this ratio, we obtain $\frac{1}{56} = \frac{1}{7} \times \frac{1}{8}$. The sum of the number of teeth in either pair of gears is $1\,(7 + 1) \times (8 + 1) = 72$. $\frac{72}{7+1} = 9$ = number of teeth in one pinion. The number of teeth in the mate for this pinion is $72 - 9 = 63$. $\frac{72}{8+1} = 8$ = number of teeth in the other pinion. The number of teeth in the mate for this pinion is $72 - 8 = 64$. As a check of the accuracy of this solution, the following proof may be employed: $9 + 63 = 8 + 64 = 72$, which is the sum of the number of teeth in each pair of gears; and $\frac{9}{63} \times \frac{8}{64} = \frac{1}{56}$, which is the required ratio for the back-gear.

The following explains the use of the method in a case where it is necessary to employ a fractional value for the factor K in order to obtain an integral value for the sum of the number of teeth in either pair of gears:

Example 2: Suppose the required ratio of gears is $\frac{1}{17}$. To factor this fraction, it is necessary to multiply and divide by some number, say 5; and using $2\frac{1}{2}$ as the value of K, the sum of the number of teeth in each pair of gears is found to be: $2\frac{1}{2}\left(\frac{17}{5} + 1\right) \times (5 + 1) = \frac{5}{2} \times \frac{22}{5} \times 6 = 66$ and $\frac{66}{\frac{22}{5}} = 15$ = number of teeth in one pinion. The number of teeth in the mate of this pinion is $66 - 15 = 51$. $\frac{66}{6} = 11$ = number of teeth in the other pinion; and $66 - 11 = 55$ = number of teeth in the mate of this pinion. As a check on the accuracy of the result, we have $15 + 51 = 11 + 55 = 66$; and $\frac{15}{51} \times \frac{11}{55} = \frac{1}{17}$, which is the required ratio.

The same method may be employed in determining the numbers of teeth in the gears of a train composed of any number of pairs of wheels, provided the center distances are the same for all pairs of wheels. Such a condition is shown at the right in Fig. 17 for a train composed of three pairs of gears, and the following example explains the use of the method in determining the numbers of teeth in each of the wheels in the transmission shown in this illustration:

Example 3: Suppose the required ratio of train is $\frac{1}{79}$. To factor, we multiply and divide by 16 and obtain $\frac{1}{4} \times \frac{1}{4} \times \frac{16}{79}$. Using the value of $\frac{16}{25}$ for the value of factor K, the sum of the numbers of teeth in each pair of wheels is found to be $\frac{16}{25}(4+1)(4+1)\left(\frac{79}{16}+1\right) = 95$. Then following the method of procedure already explained in Example 2, we find the following values for the three pairs of gears which compose the train: $\frac{19}{76}$, $\frac{19}{76}$ and $\frac{16}{79}$.

In case gears of different pitches are to be used for the different pairs of wheels in a train, the numbers of teeth may be calculated by the above method by first assuming the same pitch for all of the gears in the train. For the required pitch, the number of teeth in each pair of wheels is then obtained by multiplying the previously determined value of the number of teeth by the ratio of the assumed diametral pitch to the diametral pitch which it is required to employ.

Use of Continued Fractions for Solving Change-gear Problems. Continued fractions may be used to obtain a fraction which is small and convenient to use and which has very nearly the same value as a larger and more cumbersome fraction. A continued fraction may be defined as a fraction having unity, or 1, for its numerator, and for its denominator some number plus some fraction which also has 1 for its numerator and for its denominator some number plus a fraction, etc.

If both the numerator and the denominator of the fraction $\frac{453}{1908}$ are divided by its numerator, the fraction becomes $\frac{1}{4\frac{96}{453}}$. This process may be continued by dividing the numerator and the denominator of the fraction $\frac{96}{453}$, and the same process repeated for other fractions that might be obtained. Thus,

$$\frac{453}{1908} = \frac{1}{4^{96}\!/_{453}} \qquad \frac{96}{453} = \frac{1}{4^{69}\!/_{96}} \qquad \frac{69}{96} = \frac{1}{1^{27}\!/_{69}}$$

$$\frac{27}{69} = \frac{1}{2^{15}\!/_{27}} \qquad \frac{15}{27} = \frac{1}{1^{12}\!/_{15}} \qquad \frac{12}{15} = \frac{1}{1^{1}\!/_{4}}$$

If the fractions obtained by dividing the numerators and the denominators are written down without the fractional part of the denominator, we have, in this case, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{1}$, $\frac{1}{1}$, $\frac{1}{4}$. As will be seen, the numerators are 1 in each case, and the denominators are quotients obtained by dividing the different denominators by their numerators. In this way, the continued fraction is obtained. A common method of arranging a continued fraction is as follows:

$$\frac{453}{1908} = \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}}}}$$

A continued fraction is also frequently arranged as follows:

$$\frac{1}{4+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4} \qquad (1)$$

The method of determining the different values corresponding to the various parts of a continued fraction will now be explained. All the values for the continued fraction previously given are as follows:

$$\frac{0}{1},\ \frac{1}{4},\ \frac{4}{17},\ \frac{5}{21},\ \frac{14}{59},\ \frac{19}{80},\ \frac{33}{139},\ \frac{151}{636}. \qquad (2)$$

In order to determine these values, which are called "convergents," write the fraction $\frac{0}{1}$ and then the first fraction in the continued fraction; in this case, it is $\frac{1}{4}$. Multiply the numerator of the second fraction, in (2), by the next denominator in the continued fraction, in (1), and add the numerator of the preceding fraction, in (2); thus, $4 \times 1 + 0 = 4$. Then, multiply the denominator of the second fraction, in (2), by the next denominator in the continued fraction, in (1), and add the denominator of the preceding fraction, in (2); thus, $4 \times 4 + 1 = 17$. Write the results as the numerator and denominator of a new fraction, as shown. Multiply the numerator of the fraction last found by the next denomi-

nator in the continued fraction and add the preceding numerator to form the numerator of a new fraction; thus $1 \times 4 + 1 = 5$. Do likewise with the denominators; thus, $1 \times 17 + 4 = 21$. Proceed in this manner with the remaining denominators in the continued fraction. The last fraction is equal to the original fraction when reduced to its lowest terms. The convergents following $\frac{0}{1}$ are each nearer in value to the original fraction than any preceding one.

Application of Continued Fractions. Change-gear calculations represent typical examples of the use of continued fractions. Assume that the lead-screw of a lathe has 5 threads per inch, and the fixed gear on the stud has twice as many teeth as the gear it meshes with on the spindle. A thread is to be cut having a pitch of 4.5 millimeters; what change-gears must be used?

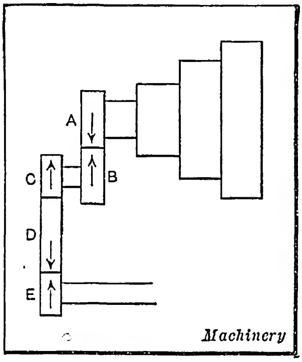

Fig. 18. Diagram illustrating Lathe Change-gears

It is assumed that the lathe is simple geared and that the arrangement of the gears is as shown by the diagram Fig. 18. It is first necessary to express the pitch of the thread to be cut and the pitch of the lead-screw in the same units; this may be done by reducing the 4.5 millimeters to inches or the pitch of the lead-screw to millimeters; the former method will be employed. Since 1 millimeter = 0.03937 inch, 4.5 millimeters = $0.03937 \times 4.5 = 0.177165$ inch. The pitch of the lead-screw is $1 \div 5 = 0.2$ inch. If gears A, B, C and E have the same number of teeth, one spindle turn will produce one turn of the lead-screw, and the tool will advance 0.2 inch. Assuming for the present that gears A and B have the same number of teeth, gear C must be smaller than gear E, since the pitch of the screw to be cut is less than that of the lead-screw, the ratio being $\frac{0.177165}{0.2} = \frac{0.885825}{1} = \frac{35{,}433}{40{,}000}$. Expressing this last fraction as a continued fraction (as described previously under the paragraph heading "Continued Fractions") the following convergents are obtained: $\frac{0}{1}, \frac{1}{1}, \frac{7}{8}, \frac{8}{9}, \frac{31}{35}, \frac{225}{254}$. The last fraction is very accurate, being equal to 0.885826; that is, when the spindle makes one turn, the lead-screw makes 0.885826 turn, and the tool advances 0.885826

$\times$ 0.2 = 0.1771652 inch. However, the terms of this fraction are too large; hence, trying the preceding one, $\frac{31}{35} = 0.885714$, and $0.885714 \times 0.2 = 0.17714+$, a value that is equal to the required pitch, within very close limits. Since gear B has twice as many teeth as gear A, gear C must have $31 \times 2 = 62$ teeth, and gear E must have 35 teeth.

Another Example: Assume that the number of threads per inch required is 14.183 and the lead-screw has 4 threads per inch. The true pitch required (disregarding tolerance) is $\frac{1}{14.183} = 0.0705$ inch. The second step is to convert the decimal into a continued fraction. Thus, $0.183 = \frac{1}{5} + \frac{1}{2} + \frac{1}{6}$, etc. Forming the various convergents, we obtain $\frac{1}{5}, \frac{2}{11}, \frac{13}{71}$, etc. Now $\frac{1}{14\frac{2}{11}} = \frac{11}{156} = 0.0705$ inch, which is the required pitch. Therefore, $\frac{4}{14\frac{2}{11}} = \frac{44}{156} = \frac{22}{78}$; hence the gear on the lead-screw must have either 156 or 78 teeth, and the gear meshing with it either 44 or 22 teeth, assuming that the fixed gears between the stud and spindle have a 1 to 1 ratio.

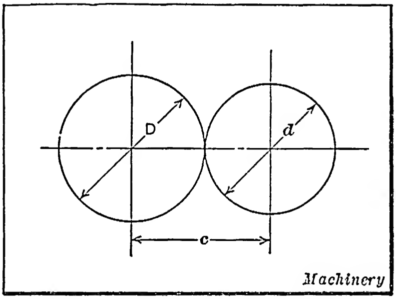

Fig. 19. Diagram for Gear Formulas

Simplified Formulas in Gearing. From Fig. 19, the number of teeth in both gear and pinion being known, then the diametral pitch of the gear is $\frac{a}{D}$ and the pinion is $\frac{b}{d}$ where,

a = number of teeth in gear (or revolutions per minute of pinion);
b = number of teeth in pinion (or R.P.M. of the gear).

The center distance c may then be expressed by either of the following two formulas:

$$c = \frac{a + b}{2\,(a \div D)} \qquad \text{or} \qquad c = \frac{a + b}{2\,(b \div d)}$$

Simplifying these expressions,

$$c = \frac{a + b}{2\,(a \div D)} = \frac{D\,(a + b)}{2a}$$

$$D\,(a+b) = 2ac \qquad D = \frac{2ac}{a+b} \tag{1}$$

$$c = \frac{a+b}{2\,(b \div d)} = \frac{d\,(a+b)}{2b}$$

$$d\,(a+b) = 2bc \qquad d = \frac{2bc}{a+b} \tag{2}$$

It will be necessary to use one of these two formulas for finding the first pitch diameter, after which, having found the value of either D or d, a simpler formula may be used, by adding Equations (1) and (2) and simplifying thus:

$$D + d = \frac{2ac + 2bc}{a+b} = \frac{2c\,(a+b)}{a+b}$$

Cancelling, $D + d = 2c$

Then, $D = 2c - d$ or $d = 2c - D$

To Solve Equations for Spiral Gearing by Trial. One of the equations used for spiral gear calculations, when the shafts are at right angles, the ratios are unequal and the center distance must be exact, is as follows:

$$R \sec \alpha + \operatorname{cosec} \alpha = \frac{2CP_n}{n}$$

In this equation R = ratio of number of teeth in large gear to number in small gear;

C = exact center distance;
P_n = normal diametral pitch;
n = number of teeth in small gear.

The exact spiral angle α of the large gear is found by trial using the equation just given.

Equations of this form are solved by trial by selecting an angle assumed to be approximately correct, and inserting the secant and cosecant of this angle in the equation, adding the values thus obtained, and comparing the sum with the known value to the right of the equal sign in the equation. An example will show this more clearly. Using the problem given in MACHINERY'S HANDBOOK, as an example, $R = 3$; $C = 10$; $P_n = 8$; $n = 28$. (This problem will be found in the section on spiral gearing and following the paragraph heading, "Shafts at Right Angles, Ratios Unequal, Center Distance Exact.")

Hence, the whole expression

$$\frac{2CP_n}{n} = \frac{2 \times 10 \times 8}{28} = 5.714$$

from which it follows that:

$$R \sec \alpha + \operatorname{cosec} \alpha = 5.714$$

In the problem given, the approximate spiral angle required is 45 degrees. The spiral gears, however, would not meet all the conditions given in the problem, if the angle could not be slightly modified. In order to determine whether the angle should be greater or smaller than 45 degrees, insert the values of the secant and cosecant of 45 degrees in the formula. The secant of 45 degrees is 1.4142, and the cosecant, 1.4142. Then,

$$3 \times 1.4142 + 1.4142 = 5.6568$$

The value 5.6568 is too small, as it is less than 5.714, which is the required value. Hence, try 46 degrees. The secant of 46 degrees is 1.4395, and the cosecant, 1.3902. Then,

$$3 \times 1.4395 + 1.3902 = 5.7087$$

Apparently an angle of 46 degrees is too small. Proceed, therefore, to try an angle of 46 degrees, 30 minutes. This angle will be found too great. Similarly 46 degrees, 15 minutes, if tried, will be found too great, and by repeated trials it will finally be found that an angle of 46 degrees, 6 minutes, the secant of which is 1.4422, and the cosecant, 1.3878, meets the requirements. Then,

$$3 \times 1.4422 + 1.3878 = 5.7144$$

which is as close to the required value as necessary.

In general, when an equation must be solved by the trial and error method, all the known quantities may be written on the right-hand side of the equal sign, and all the unknown quantities on the left-hand side. A value is assumed for the unknown quantity. This value is substituted in the equation, and all the values thus obtained on the left-hand side are added. In general, if the result is greater than the known values on the right-hand side, the assumed value of the unknown quantity is too great. If the result obtained is smaller than the sum of the known values, the assumed value for the unknown quantity is too small. By thus adjusting the value of the unknown quantity until the left-hand member of the equation with the assumed value of the unknown quantity will just equal the known quantities on the right-hand side of the equal sign, the correct value of the unknown quantity may be determined.

Analysis of Epicyclic Gear Trains. The diagrams *A* and *B*, Fig. 20, represent simple and compound epicyclic gearing and will be referred to in connection with the following method of analyzing gear trains of the epicyclic or planetary type. The two gears *a*

and b (see diagram A) are held in mesh by a link l. If this link remains stationary and if a and b represent either the pitch diameters of the gears or numbers of teeth, the revolutions of b to one turn of a equal $\frac{a}{b}\cdot$ If gear a is held stationary and link l is given one turn about the axis of a, then the revolutions of gear b, relative to link l, will also equal $\frac{a}{b}$, the same as when gear a was revolved once with the arm held stationary. Since a rotation of link l will cause a rotation of gear b in the same direction about its axis, the total number of revolutions of gear b, relative to a fixed plane, for one

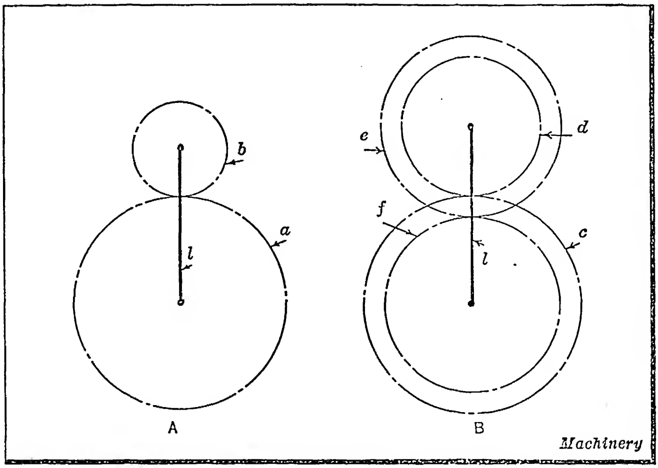

Fig. 20. Arrangement of Simple and Compound Epicyclic Gear Trains

turn of l will equal 1 (the turn of l) plus the revolutions of b relative to l, or $1 + \frac{a}{b}\cdot$ For example, if gear a has 60 teeth and gear b, 20 teeth, one turn of link l would cause b to rotate $\frac{60}{20}$, or 3 times about its own axis; gear b, however, also makes one turn about the axis of gear a, so that the total number of revolutions relative to a fixed plane equals $1 + \frac{60}{20} = 4$ revolutions.

In order to illustrate the distinction between the rotation of b around its own axis and its rotation relative to a fixed plane, assume that b is in mesh with a fixed gear a and also with an outer internal gear that is free to revolve. If the speed of the internal gear is

required, it will be necessary, in calculating this speed, to consider not only the rotation of b about its own axis, but also its motion around a, because the effect of this latter motion on the internal gear, for each turn of link l, is equivalent to an additional revolution of b.

Method of Analyzing Epicyclic Gearing. A simple method of analyzing epicyclic gearing is to consider the actions separately. For instance, with the gearing shown at A, Fig. 20, the results obtained when link l is fixed and the gear a (which normally would be fixed) is revolved are noted; if gear a is revolved in a clockwise direction, then, in order to reproduce the action of the gearing, the entire mechanism, locked together as a unit, is assumed to be given one turn counter-clockwise. The results are then tabulated, using plus and minus signs to indicate directions of rotation. Assume that gear a has 60 teeth and gear b, 20 teeth, and that $+$ signs represent counter-clockwise movements and $-$ signs clockwise movements. If link l is held stationary and gear a is turned clockwise $(-)$ one revolution, gear b will revolve counter-clockwise $(+)$ $\frac{60}{20}$ revolution. Next consider all of the gears locked together so that the entire combination is revolved one turn in a counter-clockwise $(+)$ direction, thus returning gear a to its original position. The practical effect of these separate motions is the same as though link l were revolved once about the axis of a fixed gear a which is the way in which the gearing operates normally. By tabulating these results as follows, the motion of each part of the mechanism may readily be determined:

	Gear a	Link l	Gear b
Link Stationary.......	$-$ 1 turn	0 turn	$+$ $\frac{60}{20}$ turn
Gears Locked.........	$+$ 1 turn	$+$ 1 turn	$+$ 1 turn
Number of Turns......	0	$+$ 1	$+$ 4

The algebraic sums in line headed "Number of Turns" indicate that, when gear a is held stationary and link l is given one turn about the axis of a, gear b will make 4 revolutions relative to a fixed plane in a counter-clockwise or $+$ direction, when link l is turned in the same direction.

Effect of Idler in Epicyclic Gear Train. If an idler gear were placed between gears a and b (Fig. 20), the latter would rotate about its axis in a direction opposite to that of the link, and the revolutions of gear b, relative to a fixed plane, for one turn of link l about the axis of a, would equal the difference between 1 (representing the turn of link l) and the revolutions equal to $\frac{a}{b}\cdot$ Assume

that gear a has 60 teeth, the idler gear, 30 teeth, and gear b, 20 teeth. Then the turns of b, relative to a fixed member for one turn of l about the axis of a, are shown by the following analysis:

	Gear a	Idler	Link l	Gear b
Link Stationary......	− 1 turn	$+\frac{60}{30}$ turn	0 turn	$-\frac{60}{20}$ turn
Gears Locked.......	+ 1 turn	+ 1 turn	+ 1 turn	+ 1 turn
Number of Turns....	0	+ 3	+ 1	− 2

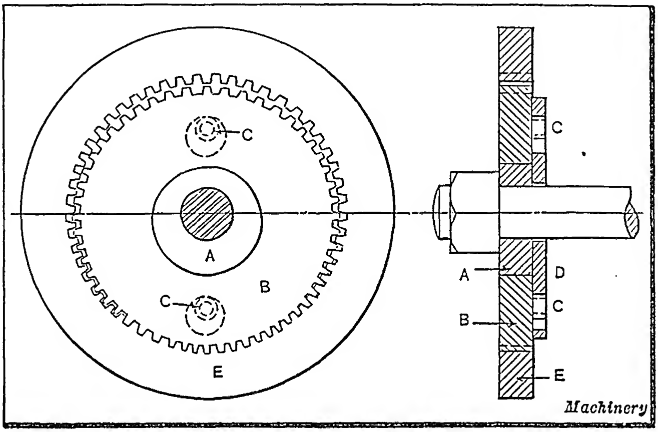

Fig. 21. Speed-reducing Mechanism

With this arrangement, the direction of rotation of b, relative to a fixed member, may or may not be in the same direction as that of the link, depending upon the velocity ratio between gears a and b. If gears a and b are of the same size, one turn of the link will cause b to revolve once about its own axis, but, as this rotation is in a direction opposite to that of the link, one motion neutralizes the other, so that b has a simple motion of circular translation relative to a fixed member. If gear b were twice as large as a, it would then revolve, for each complete turn of the link, one-half revolution about its own axis, in a direction opposite to the motion of the link; this half turn subtracted from the complete turn of the link gives a half turn of gear b in the same direction as the link, relative to a fixed member.

Compound Epicyclic Gearing. Diagram B, Fig. 20, illustrates a compound train of epicyclic gearing. This arrangement modi-

fied to suit different conditions is commonly employed. Gear *c* represents the fixed member and meshes with gear *d*, which is attached to the same shaft as gear *e*. Gear *e* meshes with gear *f* the axis of which coincides with that of fixed gear *c*. Assume that gear *c* has 36 teeth, gear *d*, 34 teeth, gear *e*, 35 teeth, and gear *f*,

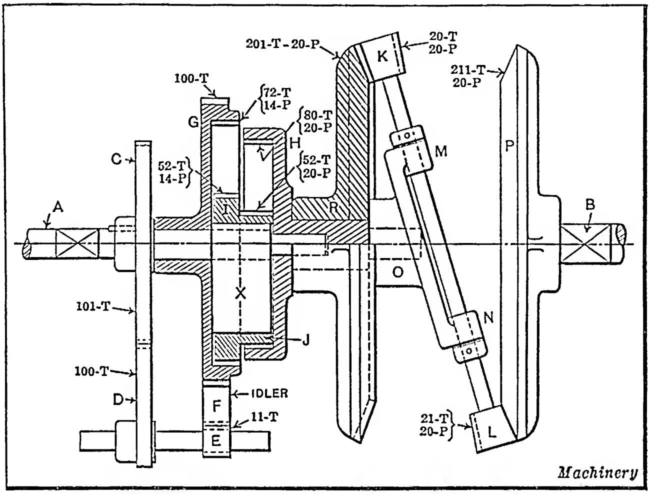

Fig. 22. Gear Combination illustrating Extreme Possibilities for Speed Reduction

35 teeth. Then one turn of link *l* about the axis of gear *c* would give the following results:

	Gear *c*	Link *l*	Gears *d* and *e*	Gear *f*
Link Stationary..	$-$ 1 turn	0 turn	$+\frac{36}{34}$ turn	$-(\frac{36}{34}\times\frac{35}{35})$ turn
Gears Locked....	$+$ 1 turn	$+$ 1 turn	$+$ 1 turn	$+$ 1 turn
Number of Turns	0	$+1$	$+2\frac{1}{17}$	$-\frac{1}{17}$

From this analysis, it will be seen that, for each counter-clockwise turn of link *l*, the rotation of gear *f* equals $1-\frac{c}{d}\times\frac{e}{f}$, in which the letters correspond either to the pitch diameters or numbers of teeth in the respective gears. If the value of $\frac{c}{d}\times\frac{e}{f}$ is less than 1, gear *f* will revolve in the same direction as link *l*, whereas, if this value is greater than 1, gear *f* will revolve in the opposite direction. Compound epicyclic gearing may be used for obtaining a very great reduction in velocity.

Ratio of Speed-reducing Mechanism. The speed-reducing mechanism shown in Fig. 21 was designed for an electric starting motor for automobile engines, to obtain a large speed reduction between the starting motor and the engine. An eccentric A, carried by the driving shaft of the motor, imparts motion to a gear wheel B which is prevented from turning by two rollers C that slide in slots in a fixed plate D. The oscillatory motion that is transmitted to the gear B imparts a much reduced rotary motion to the driven gear E, the teeth of which mesh with those on B. On the starting motor referred to, the gear B has 54 teeth and the gear E has 56 teeth. The problem is to determine the speed ratio.

Solution: Suppose, first, that the three members A, B and E are locked together and turned in a clockwise direction through one revolution. Then, in order to verify the fact that the gear B does not rotate, but has an oscillatory motion, imagine the eccentric A to be held stationary and the gear B turned in a counter-clockwise direction through a revolution, bringing it back to its original position. During this operation, gear E will turn $\frac{54}{56}$ of a revolution in a counter-clockwise direction. Since it was first turned one revolution in a clockwise direction, it is now $1 - \frac{54}{56} = \frac{2}{56}$ or $\frac{1}{28}$ of a revolution from its original position. In other words, for the one revolution that the eccentric A has been turned, the gear E has moved ahead $\frac{1}{28}$ of a revolution, and the gear reduction is 1 to $\frac{1}{28}$ or 28 to 1.

Tabulated, the result is as follows:

	A	B	E
Gears and eccentric locked......	$+1$	$+1$	$+1$
Eccentric A stationary.........	0	-1	$-\frac{54}{56}$
	$+1$	0	$+\frac{2}{56}$ or $+\frac{1}{28}$

That is, speed of A : speed of E : : $1 : \frac{1}{28}$, or as 28 : 1

Therefore in order to move gear E through one revolution, it is necessary to give the eccentric A 28 turns.

Differential Gear Problem. The differential gearing, Fig. 22, is intended as an interesting example of the limitless possibilities for speed reduction by utilizing the differential principle, although the extreme speed reduction obtained with this particular mechanism does not represent an example from practice.

A general explanation of the mechanism will first be given. The driving shaft A carries spur gear C and the eccentric X which is keyed to the shaft. Shaft A revolves in the hubs of gears G and H, the latter gear revolving in the hub of the stationary bevel gear R. The yoke O is keyed to an extension of the hub of gear H and revolves with it.

By referring to the illustration, it will be apparent that during the period of time in which yoke O is making one complete revolution, the gears K and L will make $201 \div 20$, or $10\frac{1}{20}$ revolutions. Now, since gear L has 21 teeth, $21 \times 10\frac{1}{20}$, or $211\frac{1}{20}$ equals the number of teeth that would be required for gear P, if it were to remain stationary. But having only 211 teeth, gear P must move $\frac{1}{20}$ of a tooth for each complete revolution of the yoke O, which means that O must make 211 times 20, or 4220 revolutions to 1 of gear P.

Gears G, I, J and H compose an epicyclic or planetary train of gearing which has a ratio of 10 to 1; while the train C, D, E, and G serves to retard the motion of H, and has a ratio of 100,000 to 9999. Assume that G is stationary and that C is inoperative; then, if shaft A is revolved once, H will also be turned in the same direction, and gear I, acting in the internal gear G, reduces this movement $\frac{72}{80}$ of a turn, so that after the shaft A has revolved once, gear H has advanced but $\frac{1}{10}$ of a revolution. Now, if the movement of shaft A and of the eccentric X be disregarded, and if the gear C be revolved one turn in the same direction that A was turned, the gears D, E, and G will cause gear H to turn $\frac{101 \times 11 \times 72}{100 \times 100 \times 80}$, or $\frac{9999}{100{,}000}$ revolution, in the opposite direction.

Therefore the combined effect of these actions will cause H to move in the same direction as A, an amount represented by the difference between these ratios; that is, gear H will move $\frac{1}{10} - \frac{9999}{100{,}000} = \frac{1}{100{,}000}$ revolution, for each complete turn of shaft A. It is now apparent that the ratio of movement between B and A is $\frac{1}{100{,}000}$ of the ratio of gear P to the yoke O, or

$$\frac{1}{100{,}000} \times \frac{1}{4220} = \frac{1}{422{,}000{,}000}$$

The enormous reduction thus produced may better be understood when it is mentioned that if shaft A makes 100 revolutions

per minute it will require more than eight years of continuous operation to obtain one turn of shaft B, assuming that the mechanism is not worn out before the driven shaft makes a complete revolution.

Center Distances for a Train of Gearing. The circles in Fig. 23 represent the pitch circles of gears, the diameters being as follows: gear A, 3.382 inches; gear E, 0.441 inch; gear B, 1.650 inch: gear C, 0.322 inch. The angle $FMD = \phi$ may be any angle, but in this case it is 45 degrees. Calculate distances MD, DI, JP and PI,

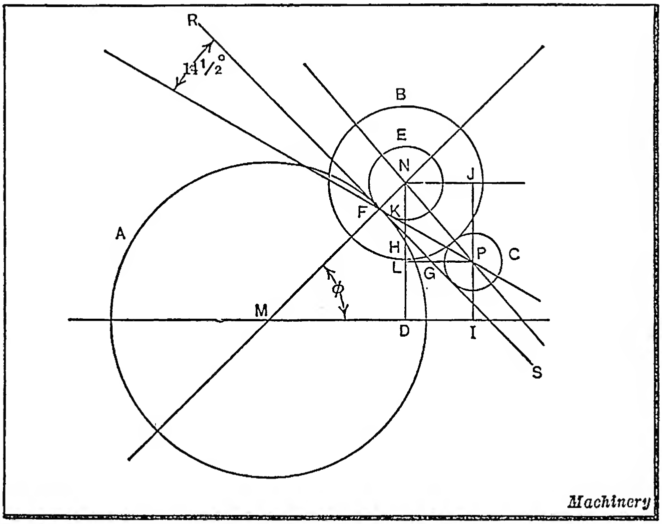

Fig. 23. Diagram illustrating Problem in Gearing

ND and JI being perpendicular to MD, and RS tangent to gears A and E.

Solution: Draw PL parallel to MD. Since RS is perpendicular to MN, angle $LHG = FMD = \phi$. Angle $LKP = LHG + GFP = \phi + 14\frac{1}{2}$ degrees $= \phi +$ 14 degrees, 30 minutes, and angle $KPL =$ 90 degrees $- (\phi +$ 14 degrees, 30 minutes) $\doteq$ 75 degrees, 30 minutes $- \phi$. Hence, when ϕ is known angle KPL is known. Angle $PFN =$ 90 degrees $-$ 14 degrees, 30 minutes $=$ 75 degrees, 30 minutes; $NF = 0.441 \div 2 = 0.2205$ inch; $NP = (1.650 + 0.322) \div 2 = 0.986$ inch; hence, in the triangle PFN, one angle and two sides are known, and the angle NPF can be found. Angle $NPL =$

$NPF + KPL$, and as NP is known, NL ($= JP$) and LP ($= DI$) are readily calculated. $MN = (3.382 + 0.441) \div 2 = 1.9115$ inch, and since the angle ϕ is known, MD and ND are readily calculated. $PI = ND - NL$. In the present case, $\phi = 45$ degrees; consequently, from the foregoing, angle $KPL = 75$ degrees, 30 minutes $- 45$ degrees $= 30$ degrees, 30 minutes. In triangle PFN, $\sin NPF = \frac{NF}{NP} \times \sin PFN = \frac{0.2205}{0.986} \times \sin 75$ degrees, 30 minutes $= 0.21651$; from which angle $NPF = 12$ degrees, 30 minutes, 14 seconds, and angle $NPL = 30$ degrees, 30 minutes $+ 12$ degrees, 30 minutes, 14

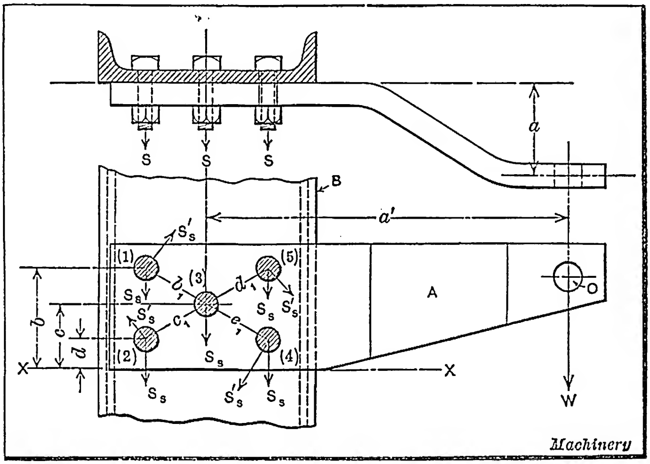

Fig. 24. Bracket attached to Channel by Bolts subjected to Stresses from Load W

seconds $= 43$ degrees, 14 seconds. $NL = NP \times \sin NPL = 0.986 \times \sin 43$ degrees, 14 seconds $= 0.67250$ inch. $LP = 0.986 \times \cos 43$ degrees, 14 seconds $= 0.72107$ inch. $MD = 1.9115 \times \cos 45$ degrees $= 1.3516$ inch. $ND = 1.9115 \times \sin 45$ degrees $= 1.3516$ inch. Finally, $LD = PI = 1.3516 - 0.6725 = 0.6791$ inch.

To Calculate Stresses in Bracket Bolts. In Fig. 24 is represented a bracket A fastened to a channel B by means of bolts or rivets. It is required to find the stresses in the fastenings produced by a load W acting at O as shown. It is apparent that the fastenings, which will be considered to be bolts, will be subjected to the following stresses:

1. The shear due to the direct action of W and indicated by the symbol S_s.

2. The shear due to the moment Wa', resisted by the moments of the bolts about their center of gravity, as a center of rotation, and represented by the symbol S'_s.

3. The tension due to the moment Wa, resisted by the moments of the bolts about edge X-X of the bracket, as the axis of rotation, and represented by the letter S.

The value of S_s may be readily found, being simply the quotient of W divided by the number N of bolts employed, that is

$$S_s = \frac{W}{N} \tag{1}$$

The values of S'_s and S are determined in the following manner: Let Q represent the stress produced in a bolt imagined to be located at unit distance from the center of rotation. Then the shearing stress S'_s in each bolt due to the moment Wa' at their actual distances is: Qb_1 for bolt (1); Qc_1 for bolt (2); 0 for bolt (3); Qe_1 for bolt (4); and Qd_1 for bolt (5). In a similar manner, the tension stress S in each bolt due to the moment Wa, with X-X as the axis of rotation, may be expressed thus: Qb for bolts (1) and (5); Qc for bolt (3); and Qd for bolts (2) and (4). It follows that the sum of the moments of these forces must equal the moments that produced them. Therefore, by first taking the moments of Qd_1, Qc_1 etc., about the center of gravity of the bolts (which obviously is the middle bolt) the equation may be written:

$$(Qb_1 \times b_1) + (Qc_1 \times c_1) + (Qd_1 \times d_1) + (Qe_1 \times e_1) = Wa'$$

$$Q = \frac{Wa'}{b_1^2 + c_1^2 + d_1^2 + e_1^2}$$

Therefore the actual stress S'_s for each bolt is the stress Q at unit distance from center of rotation, multiplied by the distance of each bolt from the middle bolt (3), thus:

$$\left.\begin{aligned}
S'_s \text{ for bolt (1)} &= Qb_1 = \frac{Wa'b_1}{b_1^2 + c_1^2 + d_1^2 + e_1^2} \\
S'_s \text{ for bolt (2)} &= Qc_1 = \frac{Wa'c_1}{b_1^2 + c_1^2 + d_1^2 + e_1^2} \\
S'_s \text{ for bolt (3)} &= Q \times 0 = 0 \\
S'_s \text{ for bolt (4)} &= Qe_1 = \frac{Wa'e_1}{b_1^2 + c_1^2 + d_1^2 + e_1^2} \\
S'_s \text{ for bolt (5)} &= Qd_1 = \frac{Wa'd_1}{b_1^2 + c_1^2 + d_1^2 + e_1^2}
\end{aligned}\right\} \tag{2}$$

By following the same method for finding the tension stress S, taking the moments of Qb, Qc, etc., about the axis X-X, the equation of moments may be expressed thus:

$$(2Qb \times b) + (Qc \times c) + (2Qd \times d) = Wa$$

$$Q = \frac{Wa}{2b^2 + c^2 + 2d^2}$$

Therefore the actual stress S for each bolt is the stress Q at unit distance from the center of rotation, multiplied by its corresponding distance from the axis X-X, thus:

$$\left.\begin{aligned} S \text{ for bolts (1) and (5)} &= Qb = \frac{Wab}{2b^2 + c^2 + 2d^2} \\ S \text{ for bolt (3)} &= Qc = \frac{Wac}{2b^2 + c^2 + 2d^2} \\ S \text{ for bolts (2) and (4)} &= Qd = \frac{Wad}{2b^2 + c^2 + 2d^2} \end{aligned}\right\} \qquad (3)$$

From the foregoing sets of formulas (2) and (3), the values of S'_s and S may be readily ascertained by observing the following rule: First ascertain the possible axis of rotation of the bracket as a whole and the center of gravity of the bolts. Then divide the moment of the weight W about this center of gravity by the sum of the squares of the distance of each bolt from the axis of rotation, and finally multiply the result by the distance to this axis (or center) from the bolt under consideration.

Since the two shearing stresses S_s and S'_s are acting in the same sectional plane, they will combine in one resultant force S_{sr} for each bolt, as shown by the diagram Fig. 25. In other words, each bolt is under the influence of but two forces, the tension force S and the resultant force S_{sr}; therefore, remembering that S and S_{sr} are acting perpendicular to each other on every particle in the cross-section of the bolts, their maximum combined effect P_1 in tension and P_2 in shear may be found by the well-known formulas for circular section bars:

$$P_1 = \tfrac{1}{2}\,(S + \sqrt{S^2 + 4S^2_{sr}}) \qquad (4)$$

$$P_2 = \tfrac{1}{2}\sqrt{S^2 + 4S^2_{sr}} \qquad (5)$$

The determination of the total combined stress to which the bolts are subjected consists simply of substituting in the foregoing fundamental formulas the respective values for tension stress and resultant shear stress.

Example Illustrating Application of Formulas. A numerical example will graphically illustrate the application of the foregoing principles. Assume that the weight W is 12,000 pounds; the diameter of the bolts ⅞ inch; and that $b = 5\frac{1}{2}$ inches, $c = 3\frac{1}{2}$ inches, $d = 1\frac{1}{2}$ inches; also b_1, c_1, d_1, and $e_1 = 6\frac{1}{4}$ inches each; $a = 4$ inches and that $a' = 10$ inches.

By substituting in Formula (1)

$$S_s = \frac{12{,}000}{5} = 2400 \text{ pounds}$$

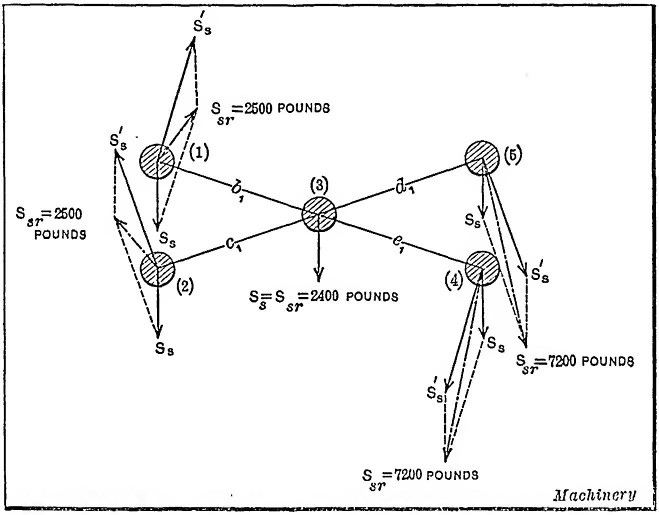

Fig. 25. Parallelogram of Force Diagrams, showing Resultant Shear Forces produced by Application of Load W in Fig. 24

and from Formula (2)

$$S'_s = \frac{12{,}000 \times 10 \times 6.25}{4 \times 6.25^2} = 4800 \text{ pounds for each bolt}$$

From the diagram Fig. 25, the resultant shearing force S_{sr} for bolts (1) and (2) = 2500 pounds; and for bolts (4) and (5) = 7200 pounds. Now, since the cross-sectional area of a ⅞-inch bolt is 0.6 square inch, the unit shearing resultant stress for bolts (1) and (2) is 2500 ÷ 0.6 = 4166 pounds per square inch; for bolt (3), 2400 ÷ 0.6 = 4000 pounds per square inch; and for bolts (4)

and (5), $7200 \div 0.6 = 12{,}000$ pounds per square inch. Having found the resultants of the two shearing forces, the tension stress S is found by substituting in formulas (3) as follows:

For bolts (1) and (5),

$$S = \frac{12{,}000 \times 4 \times 5.5}{2 \times 5.5^2 + 3.5^2 + 2 \times 1.5^2} = 3417 \text{ pounds}$$

For bolt (3),

$$S = 3.5 \times 622 = 2177 \text{ pounds}$$

For bolts (2) and (4),

$$S = 1.5 \times 622 = 933 \text{ pounds}$$

The unit tensile stress for bolts (1) and (5) is $3417 \div 0.6 = 5695$ pounds per square inch; for bolt (3), $2177 \div 0.6 = 3630$ pounds per square inch; for bolts (2) and (4), $933 \div 0.6 = 1555$ pounds per square inch.

The unit stress values for tension and shear having now been determined, the maximum unit tensile stress p_1 and unit shearing stress p_2 for each bolt may be determined by employing the fundamental equations (4) and (5). The application of these equations to bolt (1) will serve to illustrate the procedure. It will be noted that the *unit* shearing stress is substituted for S_{sr} in the equations referred to.

$$p_1 = \tfrac{1}{2}\left[5695 + \sqrt{5695^2 + 4\,(4166^2)}\right] = 7893 \text{ lbs. per sq. in.}$$

$$p_2 = \tfrac{1}{2}\sqrt{5695^2 + 4\,(4166^2)} = 5046 \text{ lbs. per sq. in.}$$

If the stresses are calculated in a similar manner for the other bolts it will be found that the bolts upon which the greatest strain is exerted are (4) and (5), while the bolt which sustains the least strain is (2), which emphasizes the importance of carefully calculating the stresses to which fastenings may be subjected.

Stresses in Brackets. In Fig. 26 is shown an ordinary bracket designed to support a load W suspended from a pin of diameter D at a distance l from the back edge of the casting. As the tendency of the bracket is to turn about point O the bolts will be subjected to tensile stress as well as shearing stress. It is clear that while the upper bolts will resist a total shearing force Q equal to that of the lower bolts, the total pulling force on each set of bolts is necessarily unequal, so that the value of P is different from that of R. The strain of the top bolts is greater than that of the lower bolts in proportion to their distances m and n from O. Therefore, if P stands for the total stress of the upper bolts, the stress R of the lower

ones will be $\frac{n}{m}P$ or $R = \frac{n}{m}P$. It is evident that to maintain equilibrium the following relation between the moments of W, P, and R about O must exist:

$$Wl = Pm + \frac{nP}{m}n = P\left(m + \frac{n^2}{m}\right) = P\left(\frac{m^2 + n^2}{m}\right)$$

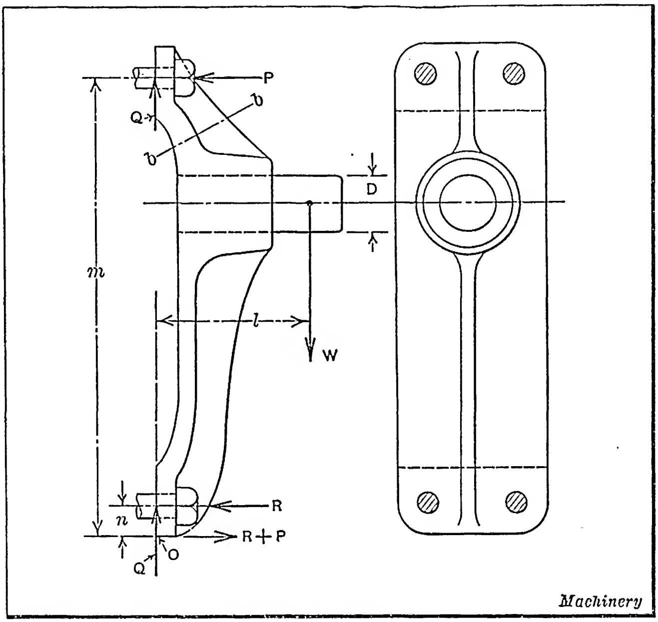

Fig. 26. Bracket Designed to Support a Weight W at a distance l from the Wall

from which it follows that:

$$P = \frac{Wl}{m^2 + n^2}m \qquad \text{and} \qquad R = \frac{n}{m}P = \frac{Wl}{m^2 + n^2}n$$

As these formulas give the tension forces acting on the bolts, the unit stress can now be determined if the diameter of the bolts is given. Since these stresses are normal to the shearing unit stresses of the forces Q, their combined effect can be determined by means of formulas:

$$P_1 = \frac{S}{2} + \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2} \qquad \text{and} \qquad P_2 = \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}$$

in which

S = unit tensile stress;
S_s = unit stress in shear;
P_1 = maximum resultant in tension;
P_2 = maximum resultant in shear.

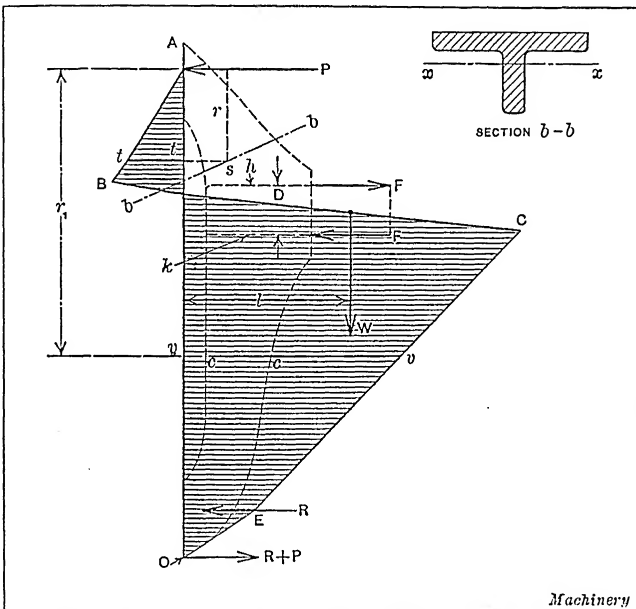

Fig. 27. Bending Moment Diagram for Bracket illustrated in Fig. 26

In order to analyze the stresses in the material of the bracket, it is necessary to draw the bending moment diagram shown in Fig. 27. It will be noted that the moment Wl in this diagram has been replaced by couple FF. (A *couple* represents two equal and parallel forces acting in opposite directions upon a body. The *arm* of a couple is the perpendicular distance between the lines of action of the forces, and the *moment* of a couple equals the product of the arm and one of the forces.) This use of a couple is necessary in order to ascertain where the bending effect of Wl on the casting

is the greatest, which depends on the location of the pin with respect to the bolts. As to the points of application of force FF it is reasonable to assume them to be applied at the top h and bottom k, respectively, of the pin having a diameter D, since it is at these places that the pin must react in order to resist the turning moment of load W. The intensity of force F is given by the equation $FD = Wl$, and is therefore equal to Wl divided by D.

The external forces can now be calculated from the foregoing equations, and as they are perpendicular to the straight line AO the bending moment diagram can be drawn. The shaded area represents such a diagram in which lines such as vv and tt, for instance, represent to a given scale the bending moments at sections c-c and b-b, respectively, of the casting, the straight line AO being the origin of the lines, and the broken line $ABCEO$, the line drawn through the end of each moment line.

Assume now that the view in the upper right-hand corner of Fig. 27 shows the section of the casting at b-b and that it is required to find the maximum stress of this section. It is first necessary to calculate the section modulus Z with respect to the neutral axis x-x, about which the resisting fibers of the material tend to revolve. If r is the perpendicular distance of the center of gravity s from the line of action of the force P, then the bending moment to which the section is subjected is $P \times r$, the value of which is represented on the diagram by line tt. The maximum stress S (when Z is the minimum value) is $S = \frac{Pr}{Z}\cdot$ From the diagram it is evident that this stress is tensile and that it occurs at the rib of the bracket.

CHAPTER VII

SPECIAL AND MISCELLANEOUS PROBLEMS

While some of the special and miscellaneous problems given in this chapter may seldom be duplicated in the experience of the average draftsman, they have been included partly to show how certain mathematical principles are applied to the unusual as well as to the more common problems. These examples are also intended to serve as a guide to the solution of other problems which, even though somewhat exceptional in character, resemble the examples in this chapter closely enough to permit employing the same general methods of solution, at least to the extent of applying the same principles.

To Determine the Rate of Production. A deceiving arithmetical problem, in shop and factory, is to find correctly how many pieces are actually being finished per hour, when the rate per hour is known for several machining operations on the same piece. One generally finds that the finished output per hour is really considerably less than what he had supposed, and there lies a mistaken idea whereby some firms actually lose money when figuring on a job. As a simple illustration, suppose a piece is being milled at the rate of 125 pieces per hour and is then drilled at the rate of 280 pieces per hour. What is the completed output per hour?

Solution: 125 pieces per hour is 1 piece in $\frac{1}{125}$ of an hour, and 280 pieces per hour is 1 piece in $\frac{1}{280}$ of an hour. Adding the fractions and reducing to the lowest terms we have, 1 piece, finished complete, in $\frac{81}{7000}$ of an hour. Then, in one hour, as many pieces can be finished complete as $\frac{81}{7000}$ are contained in 1 or $\frac{7000}{81}$ times $= 86.4$ pieces. From this we deduce the following formula in which $X =$ one rate per hour and $Y =$ the other rate per hour.

$$\text{Complete finished output per hour} = \frac{XY}{X + Y}$$

Rule: The *product* of the two rates per hour divided by the *sum* of the two rates per hour, is the complete output per hour. This is a very easy rule to remember. Also, in general, when any number of rates per hour are known, on one piece, use the same rule, treating the operations in pairs, until the final operation is performed.

To Find Width of Spline-groove Milling Cutter. For milling splineways in shafting and similar work, a milling cutter of the form

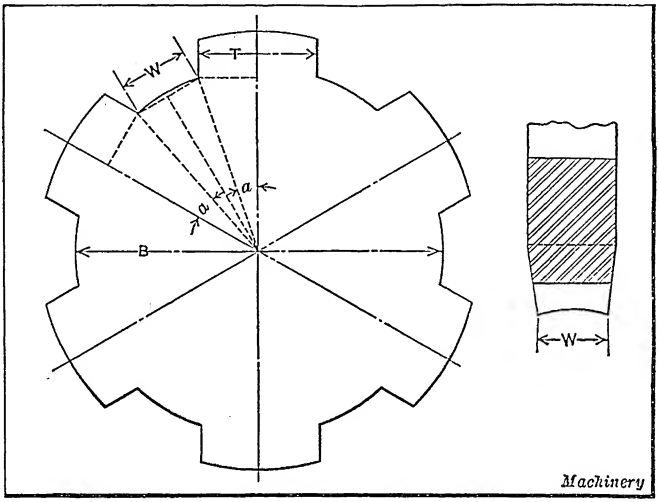

Fig. 1. To find Width W of Spline-groove Milling Cutters

shown in section in Fig. 1 may be used. The accuracy of the finished work as regards width of cut depends upon the width W of the cutting edge of the cutter. How is this width determined?

Solution: This dimension may be computed by using the following formula:

$$W = \text{Sin}\left(\frac{\frac{360 \text{ deg.}}{N} - 2a}{2}\right) \times B$$

in which N = number of splines;

B = diameter of body or of the shafting at the root of the splineway.

Angle a must first be computed, as follows:

$$\text{Sin } a = \frac{T}{2} \div \frac{B}{2} \text{ or}$$

$$\text{Sin } a = \frac{T}{B}$$

where T = width of spline;
B = diameter at the root of splineway.

This formula has been used frequently in connection with broach design, but it is capable of a more general application. If the splines are to be ground on the sides, suitable deduction must be made from dimension W to leave sufficient stock for grinding.

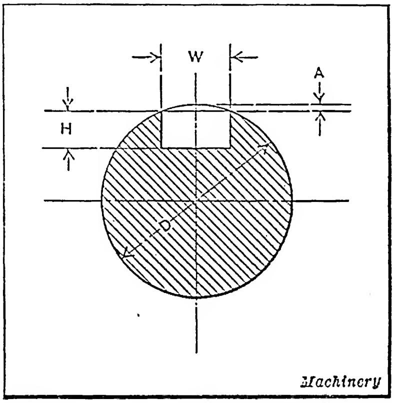

Fig. 2. To find Height A for Arc of Given Radius and Width W

To Determine Height of Arc for Keyway Milling. When milling keyways it is often desirable to know the total depth from the outside of the shaft to the bottom of the keyway. With this depth known, the cutter can be fed down to the required depth without taking any measurements other than that indicated by the graduations on the machine. In order to determine the total depth, it is necessary to calculate the height of the arc, which is designated as dimension A in Fig. 2.

Formulas: The formula usually employed to determine A for a given diameter of shaft D and width of key W, is

$$A = \frac{D}{2} - \sqrt{\left(\frac{D}{2}\right)^2 - \left(\frac{W}{2}\right)^2}$$

Another formula which is simpler than the one just given, is used in conjunction with a table of trigonometric functions as arranged in MACHINERY'S HANDBOOK. The formula follows:

$$A = \frac{D}{2} \times \text{versed sine of an angle whose cosecant is } \frac{D}{W}$$

Example: To illustrate the application of this formula, let it be required to find the height A when the shaft diameter D is $\frac{7}{8}$ inch and the width W of the key is $\frac{7}{32}$ inch. Then,

$$\frac{D}{W} = \frac{7/8}{7/32} = \frac{7}{8} \times \frac{32}{7} = 4$$

Now in a table of trigonometric functions, locate the value nearest 4 in the column headed "Cosecants," which is 3.9984. Next, in the column headed "Versed Sine," and on the same line with this cosecant, find the value 0.03178.

Then,

$$A = \frac{D}{2} \times 0.03178 = \frac{7 \times 0.03178}{8 \times 2} = 0.0139 \text{ inch}$$

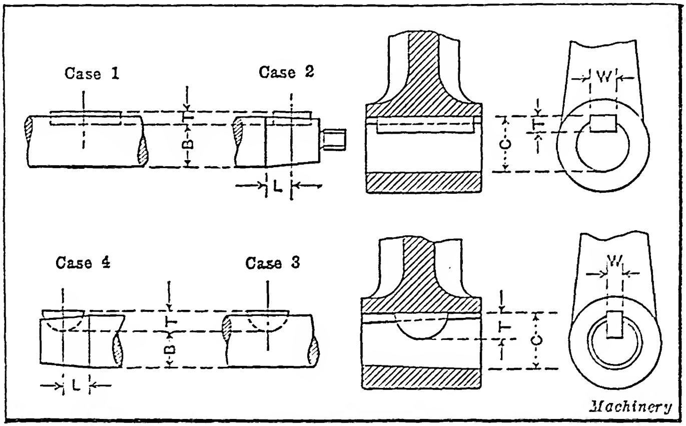

Fig. 3. Notation used in Keyway Formulas

The total depth of the keyway equals dimension H plus 0.0139 inch.

Formulas for Shaft Diameter Minus Depth of Keyseat. By means of the following formulas, the depth and other dimensions of keyways can be easily found. Some other methods give values that, having been determined, make the work of finding the depth comparatively simple; but with the formulas here given the depth can be found at once. The first formula is for a flat or feather key on a straight shaft and depth of keyway in hub as shown in Case 1, Fig. 3.

Let D = diameter of shaft;
W = width of key;
T = thickness of key;

B = diameter of shaft less depth of keyway;
$C = B + T$

$$B = \sqrt{\frac{(D - W)(D + W)}{4}} + \frac{D - T}{2} \qquad (1)$$

Example: In the case of a shaft 6 inches in diameter, what is the depth of the keyway if the key is 1.5 inches wide and 0.875 inch thick? As $D = 6$, $W = 1.5$, and $T = 0.875$, by substituting these values in the formula we get as the diameter of the shaft less the depth of the keyway:

$$B = \sqrt{\frac{(6 - 1.5)(6 + 1.5)}{4}} + \frac{6 - 0.875}{2} = 5.466$$

Subtracting this from the diameter of the shaft gives the depth of keyway $6 - 5.466 = 0.534$ inch, or ½ inch very nearly.

The second formula is for a flat or feather key on a taper shaft, as shown in Case 2.

Let D = diameter of shaft at large end;
d = diameter of shaft at center line of key;
t = taper per foot;
L = distance from large end of taper to center line of key;
W = width of key;
T = thickness of key;

$$d = D - \frac{tL}{12}$$

$$B = \sqrt{\frac{(d - W)(d + W)}{4}} + \frac{D - T}{2} \qquad (2)$$

Example: In the case of a shaft 6 inches in diameter at the large end, that has a taper of 0.5 inch per foot, where the distance from the large end of the taper to the center line of the key is 4 inches, what is the depth of the keyway, if the key is 1.5 inch wide and 0.875 inch thick? First of all it is necessary to find the diameter of the shaft at the center line of the key so as to substitute this value in Formula (2).

As $d = D - \frac{tL}{12}$, this value is $6 - \frac{0.5 \times 4}{12} = 6 - 0.1666 = 5.833$ in.

Then $B = \sqrt{\frac{(5.833 - 1.5)(5.833 + 1.5)}{4}} + \frac{6 - 0.875}{2} = 5.3805$

Subtracting the value of B from the diameter of the shaft at the large end of the keyway gives as the depth $6 - 5.380 = 0.620$ inch,

or ⅝ inch very nearly. The remaining formulas are for keyways similar to the Woodruff key. The first is for straight shafts, and the values are the same as in Formula (1).

$$B = \sqrt{\frac{(D - W)(D + W)}{4}} + \frac{W + D}{2} - T \tag{3}$$

The last is for taper shafts; the values in this case are the same as in Formula (2).

$$B = \sqrt{\frac{(d - W)(d + W)}{4}} + \frac{W + D}{2} - T \tag{4}$$

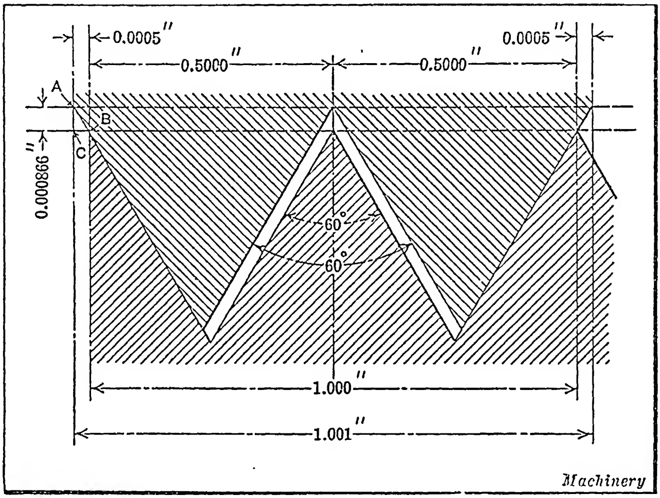

Fig. 4. Determining Amount Diameter of Nut must be increased to compensate for Lead Error

In all cases the allowance for fit for B is $\pm$ 0.001; for C from + 0.005 to + 0.007; and for W from − 0.002 to − 0.004.

To Increase Thread Diameter to Allow for Error in Lead. It sometimes happens that a screw and nut will not go together, even though the outside, pitch, and root diameters are within the required limits of accuracy, because the lead of the nut does not correspond with the lead of the screw. It is therefore necessary in some cases to increase the size of the nut or to decrease the size of the screw to compensate for the error in lead, so that the two parts can be screwed together.

Solution: The diagram, Fig. 4, illustrates a simple method of calculating the amount that the diameter of a nut must be increased to compensate for a known error in lead. It is required to find the dimension AC which is one-half the amount that the diameter of the nut must be increased over that of the screw in order to permit the two parts to be screwed together when there is an error in the lead of the nut thread of 0.001 inch per inch of thread. In the triangle ABC, side CB = 0.0005 inch and angle CAB = 30 degrees.

Therefore

$$AC = CB \times \cot CAB$$

$$AC = 0.0005 \times 1.7320 = 0.000866 \text{ inch}$$

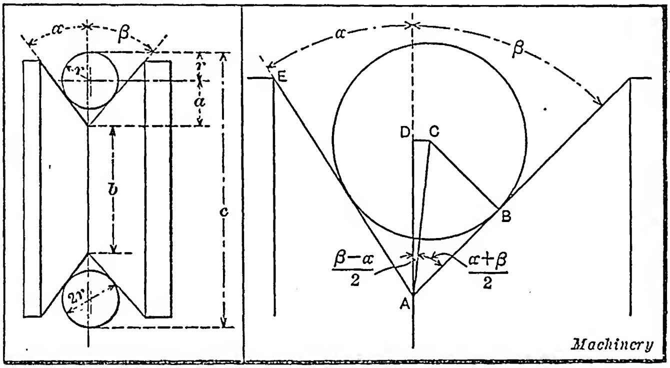

Fig. 5. To find Dimension c, having b, r and Angles α and β

As this value is for one side only, the amount which the diameter of the nut must be increased is 2 × 0.000866 = 0.001732 inch. Since the only quantities used in this solution are the differences in lead of plus or minus 0.001 inch and the thread angle of 30 degrees, it follows that the quantity 0.001732 may be employed as a constant. For instance if the error in lead should be 0.002 inch instead of 0.001 inch, the dimension AC would equal 2 × 0.000866, and the amount which the diameter of the nut thread must be increased would be 4 × 0.000866.

To Find Dimension c, Fig. 5, Having b, r and Angles α and β. A forming tool of the shape indicated at the left in Fig. 5 is to be made. Angles α and β are known, as well as the diameter of the wire, which is $2r$, and dimension b. Find dimension c in order to

insure that dimension b is correct by employing the wire method of measurement. It is assumed, of course, that the angles are accurate. In a specific example, α equals 35 degrees, β equals 40 degrees, 36 minutes; the diameter of the wires is 3/16 inch; and dimension b is 2 inches.

Solution: In order to determine the dimension c measured over the wires, first find dimension a. It is evident that $2a + 2r + b = c$. In order to determine a, draw construction lines as shown at the right in Fig. 5. Here line AD equals a. The center of the wire is at C. Line CB is at right angles to AB. Line AC, passing through the center of the circle divides angle BAE into two equal parts; hence, angle $BAC = \frac{\alpha + \beta}{2}$. Angle $CAD = \beta - \frac{\alpha + \beta}{2}$ which, simplified, may be written, $\frac{\beta - \alpha}{2}$. Further, $BC = r$.

Now, we find directly by the rules for right-angle triangles:

$AC = r \div \sin \frac{\alpha + \beta}{2}$ and

$AD = AC \times \cos \frac{\beta - \alpha}{2}$

Having thus found AD, which equals a, the problem is solved.

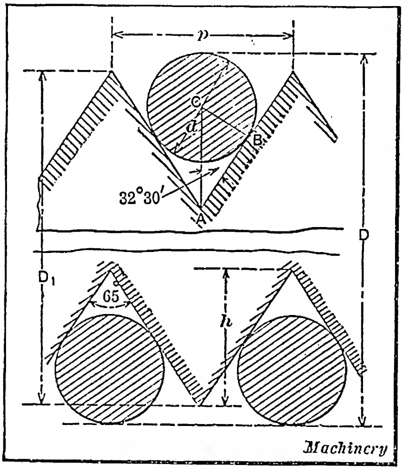

Fig. 6. To find Dimension D for a Thread of Special Angle

Inserting the given values in the formulas, we have:

$$AC = 3/32 \div \sin 37 \text{ deg. } 48 \text{ min.} = 0.15296$$
$$AD = 0.15296 \times \cos 2 \text{ deg. } 48 \text{ min.} = 0.15278 = a$$

Hence, $c = 2 \times 0.15278 + 3/16 + 2 = 2.4931$.

To Find Dimension *D*, Fig. 6, for a Special V-thread. Assume that a formula is required for determining dimension D (Fig. 6) in order to measure a 65-degree thread of sharp V-thread form by the three-wire system.

In the illustration, let:

D = diameter measured over wires;
D_1 = theoretically correct diameter over the tops of the threads;

d = diameter of wire;
h = depth of thread;
p = pitch of thread.

The dimensions D_1 and p are known; diameter d is assumed; it may be made equal to about $0.6p$; h equals $\frac{1}{2}p \times \cot 32$ degrees, 30 minutes.

By referring to the illustration, it will be seen that $D = D_1 - 2h + 2AC + d$.

But $AC = BC \div \sin 32$ degrees, 30 minutes $= \frac{1}{2}d \div \sin 32$ degrees, 30 minutes.

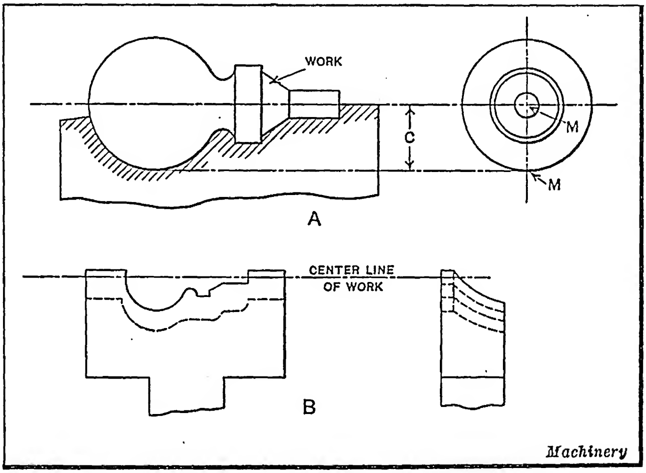

Fig. 7. (A) Example of Work. (B) Shaper Tool used for making Formed Tool for Screw Machine

Hence, $2AC = d \div \sin 32$ degrees, 30 minutes.
Inserting this value in the formula given we have:

$$D = D_1 - 2h + (d \div \sin 32 \text{ degrees, } 30 \text{ minutes}) + d$$

To Calculate Angular Position of Tools used for Planing Formed Tools. Diagram *A*, Fig. 7, represents a part to be produced in a screw machine. The dimension C is the working depth of the tool measured on a radial line. At B is shown the shaper tool employed to plane the forming tool required to produce the circular cutter. This tool has the outline of the part shown at A as determined by the radial cutting plane M–M. Fig. 8 shows the relative position of

the shaper tool and formed tool when planing the formed tool for making a circular forming tool having the cutting face in a plane other than radial, or a hook-tooth cutter. The angle a, at which the face of the shaper tool is set, equals the clearance angle b, plus the angle of correction c. For formed tools used to produce cutters having a radial cutting face, the face of the shaper tool is set at the clearance angle b only, or in a plane parallel with the plane of the formed tool face.

Calculations for Circular Forming Tools. The diagram, Fig. 9, represents a circular forming tool having its cutting face in a radial

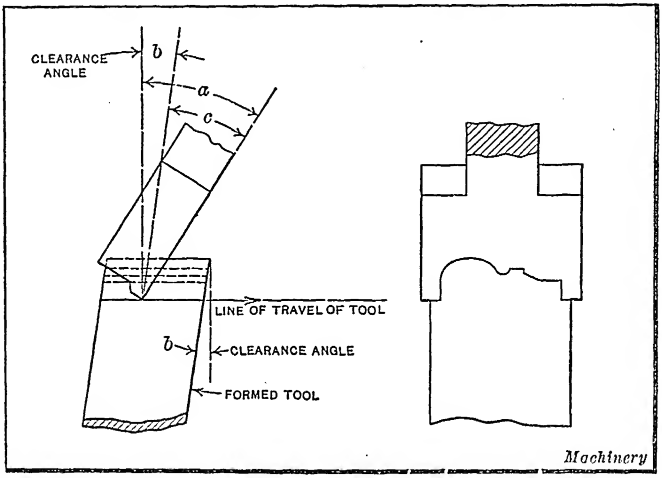

Fig. 8. Relative Positions of Shaper Tool and Formed Tool

plane of the work. Certain dimensions must be known and tabulated in all cases in order to determine the required angular position of the shaper tool, and in this case we have given the outside radius of the circular tool; the working depth of the cut, which is taken from the work specifications; the distance of the cutting face below the center of the circular tool, which varies from 1/8 to 1/2 inch, depending upon the size of the tool; and the clearance angle of the formed tool. These known quantities are represented by letters as follows:

R = outside radius of the circular tool;
C = working depth of cut (Fig. 7);

Z = distance of cutting edge below center of tool;

b = clearance angle of formed tool (Fig. 8).

The unknown dimensions on the diagram, Fig. 9, are represented as follows:

r = radial distance from center of tool to the inside of cutting form on cutting face;

V = actual length of form of formed tool, which is equal to $R - r$ and is less than the distance C.

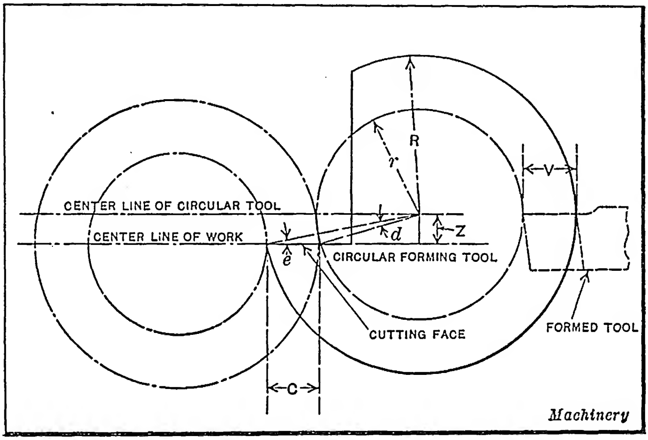

Fig. 9. Relative Positions of Work, Circular Forming Tool, and Tool used in making Forming Tool

The construction lines forming angles d and e are drawn for the purpose of calculating r.

$$\text{Sin } e = \frac{Z}{R}$$

$$\text{Tan } d = \frac{C \sin e}{R - (C \cos e)}$$

$$r = \frac{C \sin e}{\sin d}$$

$$V = R - r$$

The diagram and formula of Fig. 10 are for determining the angle a previously referred to in connection with Fig. 8. Thus in shaping a form tool for use in making a circular forming tool the shaper tool is set at an angle a which is determined by the formula,

13 P

$$\cos a = \frac{V \cos b}{C}$$

Formulas for Tools Having Top Rake. The second condition is represented by the diagram Fig. 11, which shows a circular tool having its cutting face at an angle with a radial plane of the work that passes through its highest point. The known values are tabulated as in the preceding case:

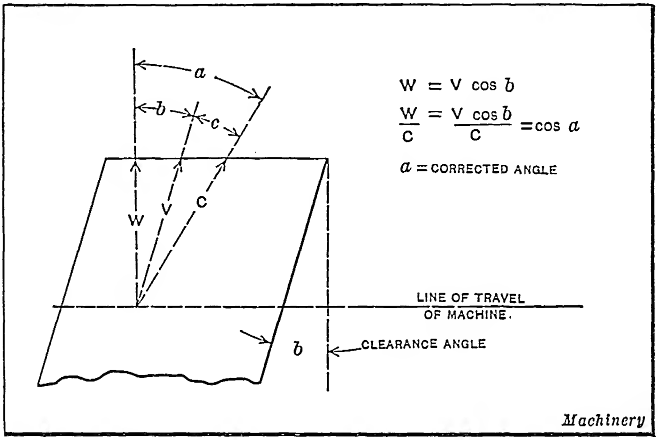

Fig. 10. Diagram showing Method of deriving Formula for determining Position of Shaper Tool

b = clearance angle of formed tool (Fig. 8);
R = outside radius of work;
r = inside radius of work = $R - C$;
e = angle of rake, or the angle that the cutting face of the cutter makes with a radial plane of the work passing through the highest point of the cutting edge;
Z = distance that highest point of cutting edge is below center of circular tool;
R_1 = outside radius of circular tool.

Construction lines are drawn as illustrated for the purpose of solving the following formulas:

$$\text{Sin } A = \frac{r \sin e}{R} \tag{1}$$

$$S = R \cos A - r \cos e \tag{2}$$

$$\operatorname{Sin} f = \frac{Z}{R_1} \tag{3}$$

$$g = f + e \tag{4}$$

$$\operatorname{Tan} d = \frac{S \sin g}{R_1 - (S \cos g)} \tag{5}$$

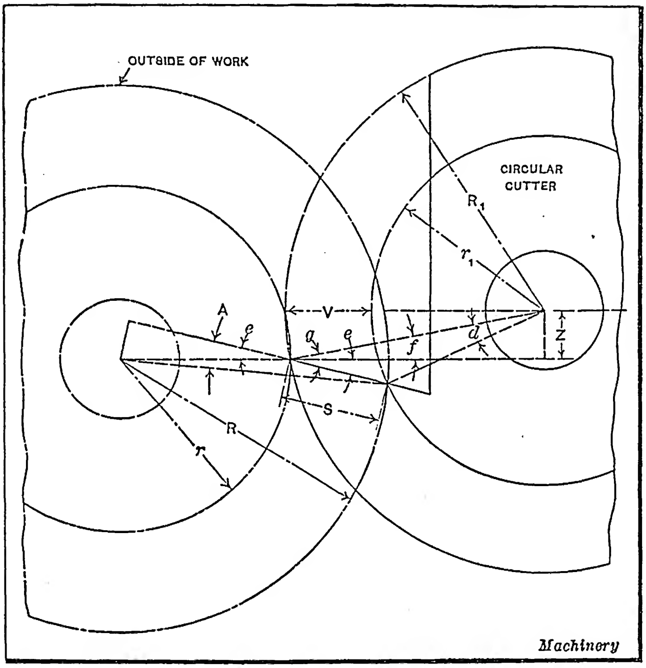

Fig. 11. Diagram representing Circular Formed Cutter having Rake

$$r_1 = \frac{S \sin g}{\sin d} \tag{6}$$

$$V = R_1 - r_1 \tag{7}$$

Again referring to Fig. 10,

$$\operatorname{Cos} a = \frac{V \cos b}{C} \tag{8}$$

Calculations for Hook-tooth Cutter. An entirely different condition is found in Fig. 12, where the relation of a hook-tooth

cutter to its formed tool is shown. In this case the face of the formed tool moves in a straight radial line toward the center of the cutter a given distance and returns to its starting position once during the rotation of the cutter through each tooth space. From the diagram, it will be seen that the working depth V of the formed tool will be less than the working depth C of the cutter due to the movement of

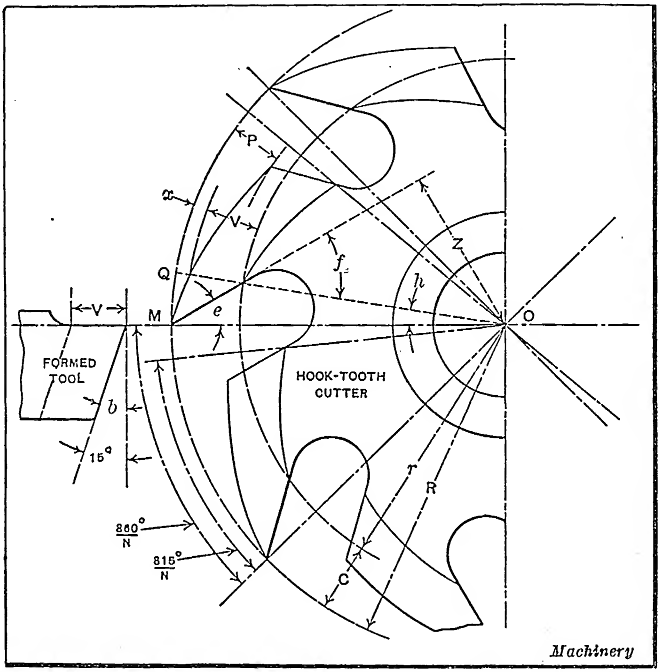

Fig. 12. Hook-tooth Cutter and Values used in determining Angle at which Shaper Forming Tool is set

the tool and the angularity of the cutting faces. The known quantities are as follows:

R = outside radius of cutter;
r = inside radius of cutter = $R - C$;
Z = distance of cutting edge behind center = $R \sin e$;
e = angle of cutter rake; $\sin e = \dfrac{Z}{R}$;
N = number of teeth in cutter;

b = clearance angle of formed tool.

P = drop of backing-off cam, or distance that formed tool moves in one direction during advance of one tooth space in rotation of cutter. (It is customary to allow ⅛ of the tooth pitch for the return movement of the backing-off tool. The tool thus has its full forward movement in ⅞ of the tooth pitch which is stated in angular terms thus: $\frac{360}{N} \times \frac{7}{8} = \frac{315}{N}$ degrees. Then $\frac{315}{N}$ degrees equals the angular movement of the cutter while the forming tool makes its extreme movement forward. If the cam used to reciprocate the formed tool is made to return in more than ⅛ of the tooth pitch, this equation must be changed to suit the condition.)

It will be seen by referring to the diagram that the formed tool starts to cut on the line OM, but does not cut on its entire form until the cutter has revolved through an angle h, when the plane represented by OQ coincides with the cutting face of the formed tool. At this stage of the cycle of movements the tool has moved toward the cutter center a distance represented by x; therefore the formed tool must have a working depth equal to V, or in other words equal to $C - x$.

Using the known quantities, we have:

$$\operatorname{Sin} f = \frac{R \sin e}{r} \tag{1}$$

$$h = f - e \tag{2}$$

$$x = \frac{NPh}{315} \tag{3}$$

$$V = C - x \tag{4}$$

In Fig. 10,

$$\operatorname{Cos} a = \frac{V \cos b}{C} \tag{5}$$

The face of the shaper tool is set at the corrected angle a when planing the formed tool form.

Example of Hook-tooth Computation. Let R = 2 inches; r = 1.5 inches; e = 15 degrees; N = 10; P = 0.200 inch; and b = 15 degrees. Substituting these values in the preceding formulas we have:

$$\operatorname{Sin} f = \frac{2 \times 0.25882}{1.5} = 0.3451 \tag{1}$$

Therefore angle f = 20 degrees 11 minutes

$$h = f - e = 20 \text{ degrees } 11 \text{ minutes} - 15 \text{ degrees} = 5 \text{ degrees } 11 \text{ minutes} \qquad (2)$$

$$x = \frac{10 \times 0.200 \times 5 \text{ degrees } 11 \text{ minutes}}{315} = 0.033 \text{ inch} \qquad (3)$$

$$V = 0.500 - 0.033 = 0.467 \text{ inch} \qquad (4)$$

$$\text{Cos } a = \frac{0.467 \times 0.96593}{0.500} = 0.9021 \qquad (5)$$

Therefore angle a = 25 degrees 34 minutes.

To Check Taper Plug Gage Diameters. The inspection of a single tapered plug may be made by taking micrometer readings over two wires of equal size, the top wires being placed on equal stacks of size blocks, and the lower wires on the same surface that the plug rests on. The problem is to determine the amount to subtract from the micrometer reading.

Solution: In the following formulas, let

T = taper per inch established by two diameters at a given distance apart;

C = amount to be subtracted from the micrometer readings in order to determine the actual diameters of the plug at the small end and at the same distance from its base as the height L of the surfaces on which the wires are resting.

Referring to the diagram Fig. 13,

a = one half the included angle of taper;

M = greater micrometer reading over wires;

m = smaller micrometer reading over wires;

G = diameter of wires; g = corresponding radius.

Then

$$T = \frac{M - m}{L} \qquad \tan a = \frac{T}{2}$$

$$\begin{aligned} X &= g \cot \tfrac{1}{2}(90 \text{ degrees} - a) \\ C &= 2g + 2X = G + 2X \\ &= G + 2g \cot \tfrac{1}{2}(90 \text{ degrees} - a) \\ &= G + G \cot \tfrac{1}{2}(90 \text{ degrees} - a) \\ &= G\,[1 + \cot \tfrac{1}{2}(90 \text{ degrees} - a)] \end{aligned}$$

The diameter of the plug at the small end and at the height L may be found by subtracting the correction C from the two measurements taken over the wires; thus

$$d = m - C \qquad D = M - C$$

$$D\,(\text{Go}) = D + hT \qquad D\,(\text{Not Go}) = D + HT$$

If the diameter of the plug is required at a point other than that at which it has already been measured, multiply the distance between the required and the measured diameter by the taper per inch, and either add the product to or subtract it from the measured diameter, according to its location. For example,

$$W = d + TY \qquad \text{or} \qquad D - TZ$$

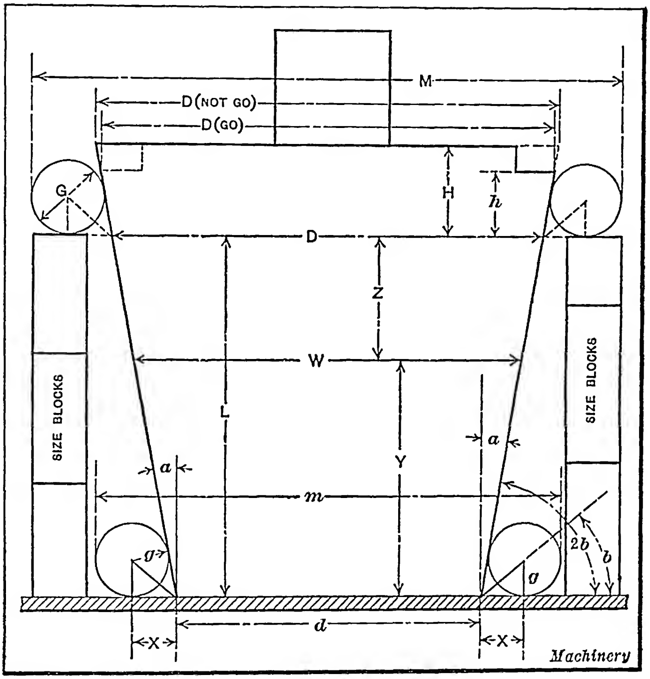

Fig. 13. Method of determining Diameters D and d for Measurements M and m over Wires of Given Diameter

To Check Taper Ring Gage by Using Taper Plug. The dimensions of a tapered ring gage such as shown in the left-hand diagram, Fig. 14, may be readily checked by means of a plug of suitable dimensions, the plug having been previously checked by measuring over wires as already described. Check the uniformity of the taper of the ring by the contact method by first making a pencil or chalk mark along the conical surface of the plug, and then de-

termining what contact the profile of the plug makes with the inner surface of the ring by observing the condition of the chalk mark after turning the plug in the ring.

The end diameters of the plug gage are known, and also the distance that it projects through the ring gage. The problem is to determine the end diameters of the ring gage.

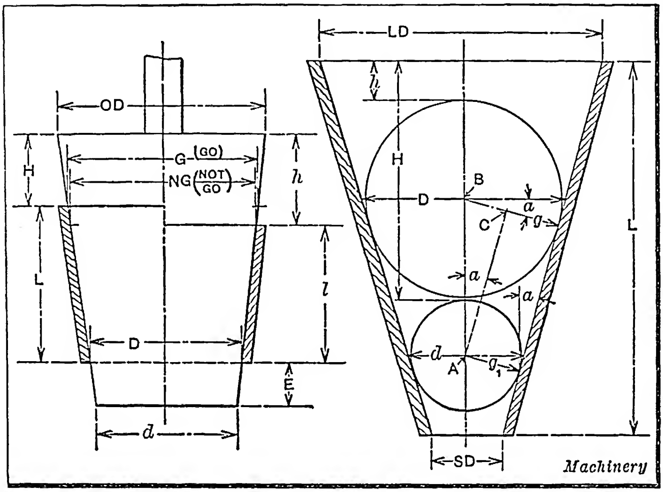

Fig. 14. Checking Tapered Ring Gages (1) by Means of Tapered Plug and (2) by using Two Balls of Given Diameters

Solution: In the formulas for determining the end diameters, let,

T = taper per inch of ring and plug;
D = diameter of plug at large end;
d = diameter of plug at small end.

Measure E, the distance that the plug extends beyond the end of the ring, using a depth micrometer.

$$D = d + ET = \text{diameter of ring at the bottom}$$
$$NG = d + T\,(l + E),\ \text{or}\ NG = OD - Th$$
$$G = d + T\,(L + E),\ \text{or}\ G = OD - TH$$

Use of Two Balls for Checking Tapered Ring Gage. The taper per inch and diameters of a tapered ring gage may be determined by dropping in two different sized steel balls of known diameter as shown by the right-hand diagram Fig. 14.

Solution: In the formulas,

G = diameter of large ball;
G_1 = diameter of small ball;
g and g_1 = the corresponding radii.

Measure the distance h, from the top of the large ball to the top of the gage, and H, the distance from the top of the small ball to the top of the gage, using a depth gage for taking both measurements.

$$AB = \text{distance between centers of balls}$$
$$= (H + g_1) - (h + g)$$
$$BC = g - g_1$$
$$\text{Sin } a = \frac{BC}{AB} = \frac{g - g_1}{(H + g_1) - (h + g)}$$
$$2 \tan a = T = \text{taper per inch}$$

The diameters D and d of the ring, which pass through the centers of the large and small balls, respectively, may then be found from the formulas

$$D = \frac{2g}{\cos a} = \frac{G}{\cos a}$$

and

$$d = \frac{G_1}{\cos a}$$

Then

$$LD = D + T\,(h + g)$$

and

$$SD = d - T\,[L - (H + g_1)]$$

Use of One Ball for Checking End Diameters of Tapered Ring Gage. A steel ball is placed inside of a taper gage as shown in Fig. 15. If the angle of the taper, length of taper, radius of ball and its position in the gage are known, how can the end diameters X and Y of the gage be determined?

Solution: The ball should be of such size as to project above the face of the gage. This, however, is not necessary, although preferable, as it permits the required measurements to be more readily obtained. After measuring the distance C, the calculation of dimension X is as follows: First obtain dimension A, which equals R divided by sin a. Then adding R to A and subtracting C we obtain dimension B. Dimension X may then be obtained by multiplying $2B$ by the tangent of angle a. The formulas for X and Y can therefore be written as follows:

$$X = 2\left(\frac{R}{\sin a} + R - C\right)\tan a$$

$$Y = X - (2T \tan a)$$

Example: If in Fig. 15 angle a = 9 degrees, T = 1.250 inches, C = 0.250 inch and R = 0.500 inch, what are the dimensions X and Y?

Applying the above formula,

$$X = 2\left(\frac{0.500}{0.15643} + 0.500 - 0.250\right) \times 0.15838$$

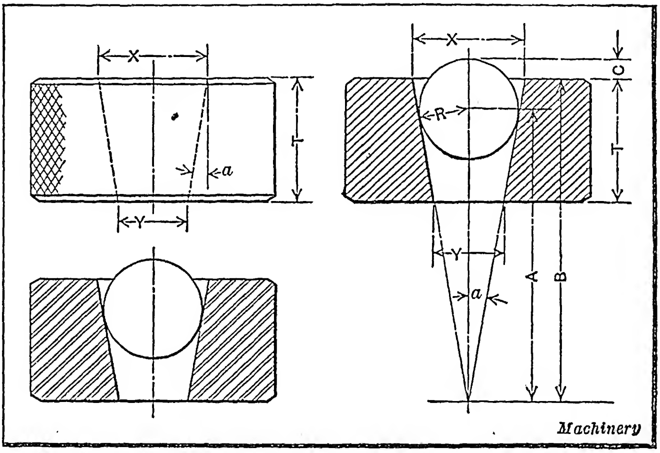

Fig. 15. Checking Dimensions X and Y by using One Ball of Given Size

Solving this equation,

$$X = 1.0917 \text{ inches}$$

Then

$$Y = 1.0917 - (2.500 \times 0.15838) = 0.6957 \text{ inch}$$

Average Pressure on Turning Tool. What is the average pressure on the tool when turning hard cast iron, taking a chip $\frac{1}{8}$ inch deep with $\frac{1}{16}$ inch feed per revolution?

The formula given by F. W. Taylor for finding the pressure on the tool is:

$$P = CD^{\frac{14}{15}}F^{\frac{3}{4}}$$

in which P = average pressure on tool in pounds;

C = a constant = 69,000 for hard cast iron;

D = depth of cut in inches;
F = feed per revolution, in inches.

Inserting the known values in this formula, we have:

$$P = 69{,}000 \times 0.125^{\frac{14}{15}} \times 0.062^{\frac{3}{4}}$$

To find the values of the two last expressions in this product, we must make use of logarithms. The whole product is also most easily found by means of logarithms.

$$\log 0.125 = \bar{1}.09691; \quad \frac{14}{15} \times \bar{1}.09691 = \bar{1}.15712$$

$$\log 0.062 = \bar{2}.79239; \quad \frac{3}{4} \times \bar{2}.79239 = \bar{1}.09429$$

$$\log\ 69{,}000 = \underline{4.83885}$$

$$\log P = 3.09026$$

Hence, $P = 1{,}231$ pounds.

Power Required for Turning. Find the average horsepower required for taking a chip in a lathe 5/16 inch deep with a feed of 5/32 inch per revolution. The material cut is a bar of 30-point carbon steel, 4 inches in diameter, and is turned at a speed of 40 revolutions per minute.

A formula for finding the horsepower for turning in a lathe, based upon the experiments of Hartig, is as follows:

$$\text{H.P.} = 0.035 \times 3.1416 \times D \times n \times d \times t \times 0.28 \times 60$$

in which H.P. = horsepower required for turning,
D = mean diameter of piece turned,
n = revolutions per minute,
d = depth of cut,
t = thickness of chip = feed per revolution.

In the problem given, D = outside diameter minus depth of cut = $4 - 5/16 = 3\,11/16$; $n = 40$; $d = 5/16$; and $t = 5/32$. If we insert these values in the given formula, we have:

$$\text{H.P.} = 0.035 \times 3.1416 \times 3.6875 \times 40 \times 0.3125 \times 0.1562 \times 0.28 \times 60 = 13.3.$$

To Find Dimension x, Fig. 16. In Fig. 16, the problem is to find the distance x between the 3/16-inch plugs.

Solution: The dimension x can be found in the following manner: In triangle OKL,

$$OK = \sqrt{1.875^2 - 0.750^2} = 1.7185 \text{ inches}$$

In triangle CKL,

$$CK = LK \times \cot 25 \text{ degrees} = 0.750 \times 2.1445 = 1.6084 \text{ inches}$$
$$OC = OK - CK = 1.7185 - 1.6084 = 0.1101 \text{ inch}$$

Draw line OB parallel to CF, making angle COD equal 25 degrees. Now,

$$FB = CD = OC \times \sin 25 \text{ degrees} = 0.1101 \times 0.42262 = 0.0465 \text{ in.}$$

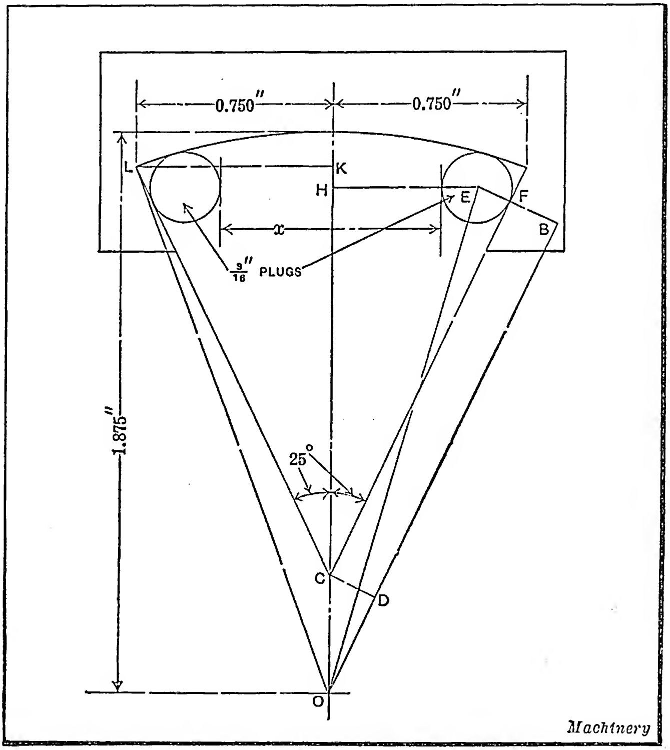

Fig. 16. To find Dimension x between Plugs of Given Diameter

and $\quad EB = EF + FB = 0.0937 + 0.0465 = 0.1402$ inch

Also, $\quad OE = 1.875 - 0.0937 = 1.7813$ inches

Therefore, $\quad \text{Sin } EOB = \dfrac{EB}{OE} = \dfrac{0.1402}{1.7813} = 0.07871$

Hence angle EOB = 4 degrees 30 minutes 50 seconds or approximately 4 degrees 31 minutes.

Now in triangle HOE, angle $HOE = COD - EOB$ = 25 degrees − 4 degrees 31 minutes = 20 degrees 29 minutes and $HE = OE \times \sin$ 20 degrees 29 minutes = $1.7813 \times 0.34993 = 0.6233$ inch

Therefore,

$$\frac{x}{2} = HE - 0.0937 = 0.6233 - 0.0937 = 0.5296 \text{ inch}$$

and

$$x = 2 \times 0.5296 = 1.0592 \text{ inches}$$

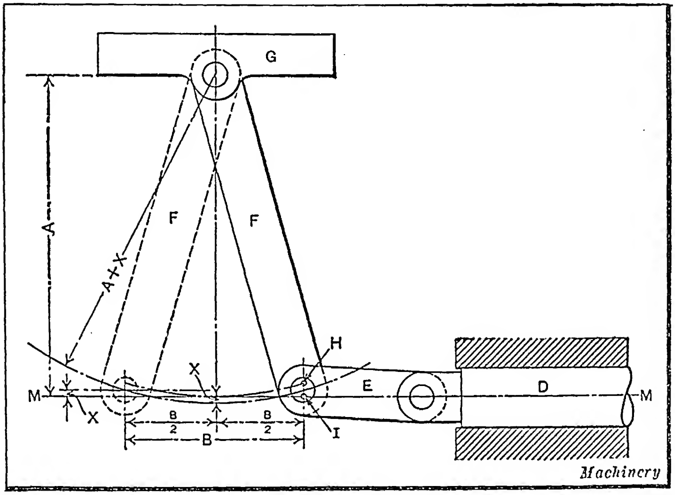

Fig. 17. Determining Length of Link *F* so that Link *E* will swing equally above and below the Center Line

To Determine Length of Link *F*, Fig. 17. In designing a motion of the type shown in Fig. 17, it is essential, usually, to have link E swing equally above and below the center line MM. A mathematical solution of this problem follows. In the illustration, G represents the machine frame; F, a lever shown in the extreme positions; E, a link; and D, a slide. The distances A and B are fixed and the problem is to obtain $A + X$, or the required length of the lever. In the right triangle:

$$A + X = \sqrt{(A - X)^2 + \left(\frac{B}{2}\right)^2}$$

Squaring, we have:

$$A^2 + 2AX + X^2 = A^2 - 2AX + X^2 + \frac{B^2}{4}$$

$$4AX = \frac{B^2}{4}$$

$$X = \frac{B^2}{16A}$$

$$A + X = A + \frac{B^2}{16A}$$

$$= \text{length of lever}$$

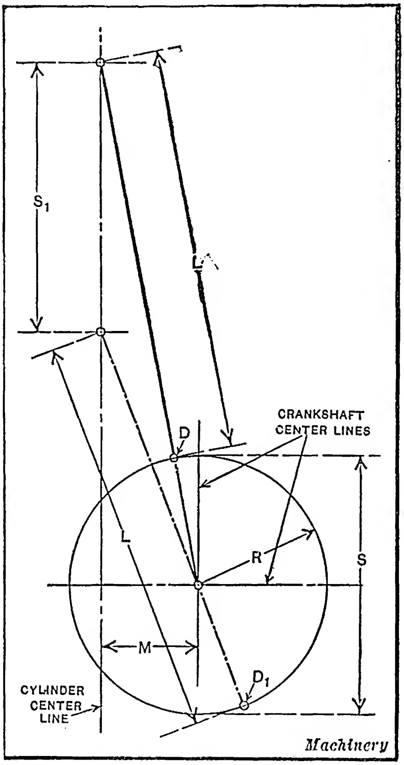

Fig. 18. To find Length of Stroke S_1 obtained with Offset Crankshaft

Length of Stroke Obtained with Offset Crankshaft. Engines that have the center of the crankshaft located on the center line of the cylinder have a piston stroke which is exactly equal to the crankshaft stroke. On engines, the design of which provides an offset crankshaft, where the center lines of the crankshaft and cylinder do not coincide, the crank and piston strokes are not equal, the piston stroke being greater than that of the crankshaft. In Fig. 18, S represents the crankshaft stroke, S_1 the piston stroke, L the length of the connecting-rod, M the offset of the crankshaft center, and R the throw of the crank. Centers D and D_1 represent the upper and lower dead centers of the crankshaft wrist-pin, and the full and dotted lines designated by the dimensions L show the respective positions of the connecting-rod at the end of each stroke of the piston.

Then,

$$S_1 = \sqrt{(L + R)^2 - M^2} - \sqrt{(L - R)^2 - M^2}$$

Example: Assume $L = 10$ inches, $R = 2\frac{1}{2}$ inches, $S = 5$ inches, and $M = 1\frac{1}{2}$ inches. From the formula,

$$S_1 = \sqrt{(10 + 2.5)^2 - 1.5^2} - \sqrt{(10 - 2.5)^2 - 1.5^2}$$
$$= \sqrt{12.5^2 - 1.5^2} - \sqrt{7.5^2 - 1.5^2}$$
$$= 12.4097 - 7.3485 = 5.0612 \text{ inches}$$

Therefore, S_1 is greater than S.

Length of Hole in Floor for Flywheel. A flywheel is 16 feet in diameter (outside measurement) and the center of its shaft is 3 feet above the floor; how long must the hole in the floor be to let the flywheel turn?

Solution: The conditions are as represented in Fig. 19. The line AB is the floor level and is a chord of the arc ADB; it is parallel to the horizontal diameter through the center O. CD is a vertical diameter and is perpendicular to AB. It is shown in geometry that the diameter CD bisects the chord AB at the point of intersection E. Now, one of the most useful theorems of geometry is that when a diameter bisects a chord, the product of the two parts of the diameter is equal to the square of one half the chord; in other words, $AE^2 = ED \times EC$. If AB is represented by L and OE by a, $ED = r - a$ and $EC = r + a$, in which r = the radius OC; hence,

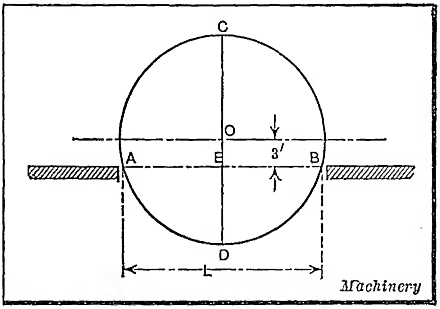

Fig. 19. To find Length of Hole in Floor for Flywheel

$$\left(\frac{L}{2}\right)^2 = (r - a)(r + a) = r^2 - a^2$$

and

$$L = 2\sqrt{r^2 - a^2}$$

Substituting the values given,

$$L = 2\sqrt{8^2 - 3^2} = 14.8324 \text{ feet} = 14 \text{ feet, } 10 \text{ inches}$$

The length of the hole should be at least 15 feet, to allow for clearance.

Proportioning Cone Pulley Steps to Give Constant Belt Tension. The problem is to calculate the diameters of the unknown steps on the larger of a pair of cone pulleys so that the tension on an open belt will be the same on all steps. The diameters of the steps of the smaller pulley are 3, 4, 5, and 6 inches, respectively, and the

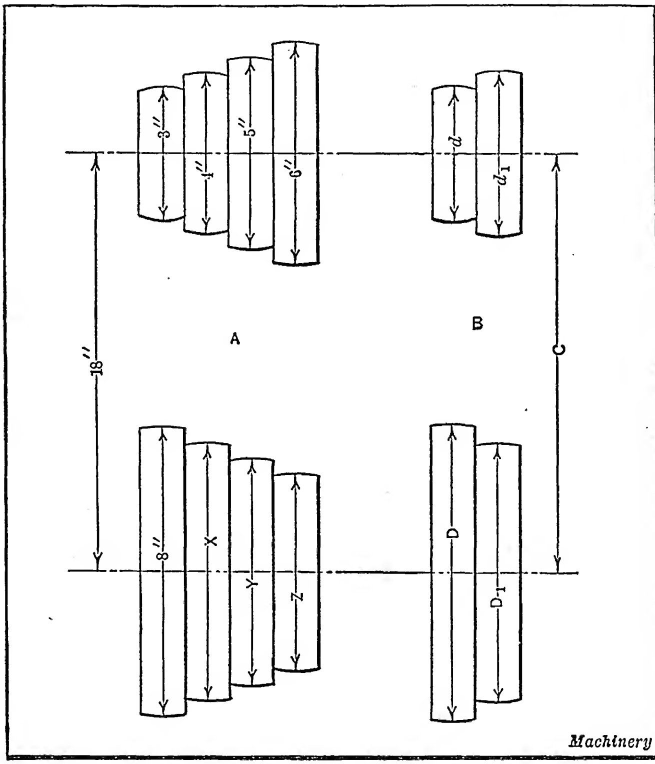

Fig. 20. Diagram representing Problem in Cone Pulley Design

largest step of the large pulley is 8 inches in diameter. The distance between the shaft centers is 18 inches. Fig. 20 shows diagrammatically at *A* the conditions of the problem, dimensions *X*, *Y* and *Z* being unknown.

Solution: The length of belt required to pass over two pulleys is obtainable by a formula which follows:

$$L = 2C + \frac{11D + 11d}{7} + \frac{(D-d)^2}{4C} \tag{1}$$

in which L = length of belt;
D = diameter of large step;
d = diameter of small step; and
C = center distance between pulleys.

By substituting $\frac{\pi}{2}$ for $\frac{11}{7}$, this formula may be written as follows:

$$L = 2C + \frac{\pi}{2} \times (D + d) + \frac{(D-d)^2}{4C} \tag{2}$$

From this a formula can be derived for finding the diameter of any unknown step on either pulley that will cause the belt tension to be the same on any pair of steps. This is, of course, due to the fact that the proper belt length for any pair of steps will be constant. In the following formulas, L and C equal the same values as before, and

D = largest step on larger pulley;
d = smallest step on smaller pulley;
D_1 = any step on larger pulley; and
d_1 = corresponding step on smaller pulley.

Then, from Formula (1) and diagram B in the illustration,

$$L = 2C + \frac{\pi}{2}(D_1 + d_1) + \frac{(D_1 - d_1)^2}{4C}$$

Equating this with Formula (2)

$$2C + \frac{\pi}{2}(D + d) + \frac{(D-d)^2}{4C} = 2C + \frac{\pi}{2}(D_1 + d_1) + \frac{(D_1 - d_1)^2}{4C}$$

Arranging according to powers of $D_1 - d_1$,

$$(D_1 - d_1)^2 + 2\pi C(D_1 - d_1) + 4C\left[\pi d_1 - \frac{\pi}{2}(D + d)\right] - (D - d)^2 = 0$$

Solving this quadratic equation for $D_1 - d_1$ and taking the plus sign before the radical,

$$D_1 - d_1 = \sqrt{(\pi C + D + d)^2 - 4(\pi C d_1 + Dd)} - \pi C \tag{3}$$

From this formula D_1 or d_1 can be found when either one is known. Substituting the known values

$$\begin{aligned} D_1 &= \sqrt{(18\pi + 8 + 3)^2 - 4(18\pi d_1 + 8 \times 3)} - (18\pi - d_1) \\ &= \sqrt{4562.8404 - 226.1952 d_1 - 96} - (56.5488 - d_1) \\ &= \sqrt{4466.8404 - 226.1952 d_1} - (56.5488 - d_1) \end{aligned}$$

Now solving for diameter X in the illustration, in which case $d_1 = 4$ inches,

$$X = \sqrt{4466.8404 - 226.1952 \times 4} - (56.5488 - 4)$$
$$= \sqrt{3562.0596} - 52.5488$$
$$= 59.683 - 52.549 = 7.134 \text{ or } 7\tfrac{1}{8} \text{ inches}$$

Solving for Y in which case $d_1 = 5$ inches,

$$Y = \sqrt{4466.8404 - 226.1952 \times 5} - (56.5448 - 5)$$
$$= \sqrt{3335.8644} - 51.5448$$
$$= 57.757 - 51.549 = 6.208 \text{ or } 6\tfrac{13}{64} \text{ inches}$$

Finally, solving for Z, in which case $d_1 = 6$ inches,

$$Z = \sqrt{4466.8404 - 226.1952 \times 6} - (56.5488 - 6)$$
$$= \sqrt{3109.6692} - 50.5488$$
$$= 55.764 - 50.549 = 5.215 \text{ or } 5\tfrac{7}{32} \text{ inches}$$

To Determine Length of Rolled Belt. The length, in feet, of a closely-rolled belt may be determined by the following rule: Find the sum of the outside diameter and the diameter of the hole in inches. Multiply this sum by the constant 0.1309, and the result by the number of complete turns of the belt. The rule is based on the rule for obtaining the length of a plane spiral. In order to obtain the length of the spiral, the sum of the inner and outer diameters is divided by 2, and the quotient is multiplied by 3.1416. The product is again multiplied by the number of turns made by the spiral. In the application of this rule, if the inner and outer diameters are in inches, the length of the spiral will also be in inches. If instead of dividing by 2 and multiplying by 3.1416, the sum of the inner and outer diameters is multiplied by the constant, 0.1309, as previously mentioned, the result will be obtained in feet, since $3.1416 \div (2 \times 12) = 0.1309$.

Another method of determining the length follows: By means of a yardstick or rule, measure the distance $(t + d)$ which amounts to the total thickness of the ring of belting material plus the diameter of the hole. Taking the measurement $(t + d)$ in inches, multiply it by 0.2618, and multiply the product thus obtained by the number of complete turns in the roll. The result is the length L of the belt in feet. Expressed as a formula,

$$L = 0.2618\,(t + d)\,N$$

where t = total thickness, in inches, of ring of belting;
d = diameter, in inches, of opening or hole; and
N = number of complete turns.

The measurement $(t + d)$ can often be taken without removing the roll of belting from its place in the stock-room. It will be noted that the method here given requires only one measurement.

Diameter of Steel Ball for Given Weight. If a steel ball 0.718 inch diameter weighs 380 grains, what must the diameter of a ball be to weigh 383 grains?

Solution: The cubic contents of spheres are to one another as the cubes of their diameters. Therefore, the relation between two balls is expressed by the proportion $x^3 : y^3 = a : b$, in which x and y represent the diameters of the balls and a and b the given weights.

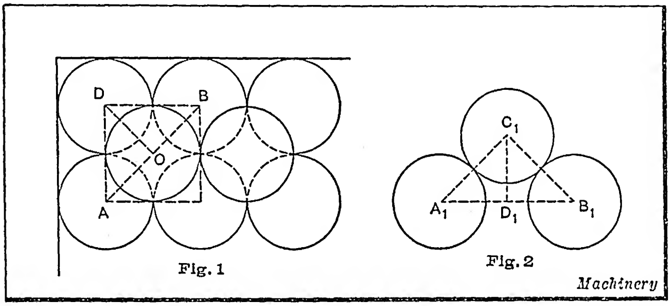

Fig. 21. To find Number of Balls that can be placed in a Cubical Box

From this we obtain the expression $x^3 : 0.718^3 = 383 : 380$; clearing for x yields

$$x = \sqrt[3]{\frac{0.718^3 \times 383}{380}} = 0.720 \text{ inch},$$

the required diameter.

Number of Balls that can be Placed in a Cubical Box. How many balls, each 1 inch in diameter, can be placed in a box 12 inches by 12 inches by 12 inches, inside measurements?

Solution: If the balls are arranged in layers, as indicated in Fig. 21, it is evident that $12 \times 12 = 144$ balls will constitute the first layer and $11 \times 11 = 121$ balls, the second layer. The distance between the plane of centers of the first layer and the plane of centers of the second layer is represented by the altitude C_1D_1 of the triangle $A_1C_1B_1$. The side A_1B_1 is equal to AB, and as $DA = DB = 1$ inch, $AB = A_1B_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$ inches. $A_1C_1 = B_1C_1 = 1$ inch. Hence, $C_1D_1 = \sqrt{1^2 - (\frac{1}{2}\sqrt{2})^2} = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2} = 0.707$

inch. The centers of the first layer of balls must be ½ inch from the bottom of the box, and the centers of the top layer must be at least this distance from the top of the box. Therefore, the number of layers must be at least equal to $\frac{12 - 0.5 - 0.5}{0.707} = 15$, to the nearest integer. If there are fifteen layers, eight will contain 144 balls each and seven will contain 121 balls each. The distance between the plane of centers of the first layer and the plane of centers of the fifteenth layer is $14 \times 0.707 = 9.898$ inches. The distance from the bottom of the box to the plane tangent to the tops of the balls

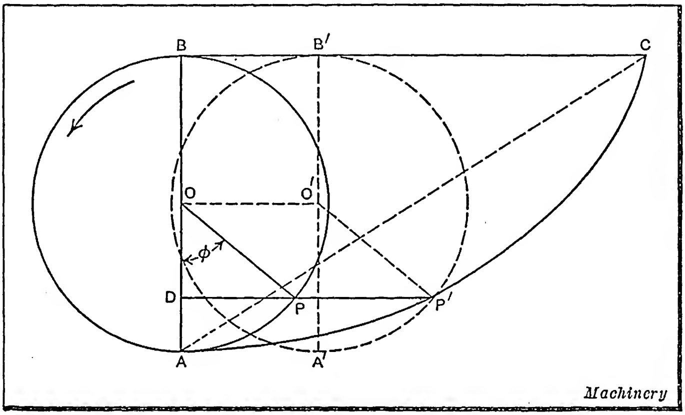

Fig. 22. Curve of Quickest Descent

in the fifteenth layer is $9.898 + 0.5 + 0.5 = 10.898$ inches. Consequently, another layer of 144 balls can be laid on top of the fifteenth layer, thus making nine layers of 144 balls each and seven layers of 121 balls each. Hence, the total number of balls that can be placed in the box is $9 \times 144 + 7 \times 121 = 2143$ balls.

Line of Quickest Descent. Suppose that a ball is to roll without friction from a point C to a point A, Fig. 22; what must be its path in order to get to A in the shortest time?

If the ball rolls from C to A along the straight inclined plane CA, the acceleration will be uniform throughout the entire distance. If, however, the ball takes a curved path, such as $CP'A$, the acceleration will be variable, and a velocity that nearly approaches the final velocity at A will be attained for some time before the ball reaches A. It will thus be apparent that there must be some curve

that will take the ball from C to A in a shorter time than if it follows a straight line. The curve is called the brachistochrone, a word derived from the Greek and meaning shortest time. It is commonly called the cycloid, and is the curve that would be traced by a point on the circumference of a circle rolling on a straight line without slipping.

Assume that a circle having the diameter BA rolls from B toward C, turning in the direction indicated by the arrow; the path (locus) of the point A will be the half cycloid $AP'C$, and BC will be equal in length to half the circumference of the circle. For any position of the circle, as B', the center will be at O'; the point A will be at P'; and $BB' = \text{arc } AP = OO'$. To construct the cycloid, divide the semicircle APB into any convenient number of parts; suppose that one of these parts is AP. Through P draw DP' parallel to BC; draw OO', and make it equal in length to the arc AP. With O' as a center and OB as a radius, describe a circle cutting DP' in P'. Note that angle $A'O'P' = AOP$ and that $POO'P'$ is a parallelogram; hence, $PP' = OO' = \text{arc } AP$; $DP = r \times \sin\phi$; and $DA = r\,(1 - \cos\phi)$. Also, $DP' = r\,(\sin\phi + \phi)$, in which ϕ is the angle AOP, in radians. Therefore, by using a table of trigonometric functions, the values of DA and DP' may be readily calculated for any angle desired.

Periphery of the Ellipse. Many formulas have been printed concerning the periphery of the ellipse, all of which are practically worthless when the ratio of the axes exceeds 8 or 9 to 1. The only exact expression for the periphery is by means of formulas that contain an infinite series, and no simple expression for the sum of these series has yet been derived. In order better to understand the nature of the difficulty, let:

p = periphery;

a = semi-major axis;

b = semi-minor axis;

D = long diameter $= 2a$;

d = short diameter $= 2b$;

n = ratio of difference and sum of axes $= \dfrac{D-d}{D+d} = \dfrac{a-b}{a+b}$;

r = ratio of axes $= \dfrac{D}{d} = \dfrac{a}{b}$;

k = a constant.

Then:
$$p = \pi\,(a+b)\,k = \frac{\pi}{2}\,(D+d)\,k \tag{1}$$

Ratios n and Corresponding Values of k

n	k	n	k	n	k	n	k
0	1	0.25	1.0157	0.50	1.0635	0.75	1.1465
0.01	1.0000	0.26	1.0170	0.51	1.0662	0.76	1.1506
0.02	1.0001	0.27	1.0183	0.52	1.0688	0.77	1.1548
0.03	1.0002	0.28	1.0197	0.53	1.0716	0.78	1.1591
0.04	1.0004	0.29	1.0211	0.54	1.0743	0.79	1.1634
0.05	1.0006	0.30	1.0226	0.55	1.0772	0.80	1.1678
0.06	1.0009	0.31	1.0242	0.56	1.0801	0.81	1.1723
0.07	1.0012	0.32	1.0258	0.57	1.0830	0.82	1.1768
0.08	1.0016	0.33	1.0274	0.58	1.0860	0.83	1.1815
0.09	1.0020	0.34	1.0291	0.59	1.0891	0.84	1.1862
0.10	1.0025	0.35	1.0309	0.60	1.0922	0.85	1.1909
0.11	1.0030	0.36	1.0327	0.61	1.0954	0.86	1.1958
0.12	1.0036	0.37	1.0345	0.62	1.0987	0.87	1.2007
0.13	1.0042	0.38	1.0364	0.63	1.1020	0.88	1.2057
0.14	1.0049	0.39	1.0384	0.64	1.1053	0.89	1.2108
0.15	1.0056	0.40	1.0404	0.65	1.1088	0.90	1.2160
0.16	1.0064	0.41	1.0425	0.66	1.1123	0.91	1.2213
0.17	1.0072	0.42	1.0446	0.67	1.1158	0.92	1.2266
0.18	1.0081	0.43	1.0468	0.68	1.1194	0.93	1.2321
0.19	1.0090	0.44	1.0490	0.69	1.1231	0.94	1.2376
0.20	1.0100	0.45	1.0513	0.70	1.1268	0.95	1.2433
0.21	1.0111	0.46	1.0536	0.71	1.1306	0.96	1.2490
0.22	1.0121	0.47	1.0560	0.72	1.1345	0.97	1.2549
0.23	1.0133	0.48	1.0585	0.73	1.1384	0.98	1.2609
0.24	1.0145	0.49	1.0610	0.74	1.1424	0.99	1.2670
0.25	1.0157	0.50	1.0635	0.75	1.1465	1.00	1.2732

$$k = 1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1}{2 \times 4}\right)^2 n^4 + \left(\frac{1 \times 3}{2 \times 4 \times 6}\right)^2 n^6$$
$$+ \left(\frac{1 \times 3 \times 5}{2 \times 4 \times 6 \times 8}\right)^2 n^8 + \text{etc.} \tag{2}$$

Before discussing this series, it will be well to establish the relation between r and n, which is easily done; thus:

$$\frac{a-b}{a+b} = \frac{\dfrac{a}{b} - 1}{\dfrac{a}{b} + 1} = n = \frac{r-1}{r+1} \tag{3}$$

From Equation (3):

$$r = \frac{1 + n}{1 - n} \tag{4}$$

The limits of r are 1 and ∞, and the limits of n are 0 and 1; in other words, as n increases from 0 to 1, r increases from 1 to ∞. Thus, from Equation (3), for $r = 1$, $n = 0$, and from Equation (4), for $n = 1$, $r = \infty$. Moreover, comparatively small values of r give values of n that approach 1; and the nearer n is to 1, the greater must be the number of terms used in Equation (2) to obtain an accurate value of k. For example, for $n = 0.90$, $r = 19$, that is, the ratio of the axes is 19 to 1; and to obtain k correct to five significant figures, it will be necessary to use about twenty terms of the series. When calculating the accompanying table "Ratios n and Corresponding Values of k," the series was extended (the law of which is evident) until the last term used gave a value less than 0.000000005, the result being to obtain the value of k to eight decimal places, with a possible error of one unit in the last place, but guaranteeing the correctness of the figure in the seventh place. For $n =$ 0.98, it was necessary to use eighty-four terms of the series. But when $n = 0.98$, $r = 99$; hence when n increases from 0.98 to 1, r must increase from 99 to ∞. In fact, if n exceeds, say, 0.70 (for which $r = 5\frac{2}{3}$), it becomes increasingly difficult to obtain an accurate value for k.

When $r = 1$, the axes are equal, and the ellipse becomes a circle; n is then 0, and by Formula (2), $k = 1$. When $r = \infty$, $b = 0$, and the ellipse becomes a right line, the periphery being equal to twice the major axis, or $4a$. Substituting in Formula (1),

$$k = \frac{4}{\pi} = 1.2732395 +$$

Therefore, the limits of k are: for $n = 0$, $k = 1$; and for $n = 1$, $k = 1.2732395 +$. A little consideration of the limits of r, n, and k will make it evident why it is so difficult to obtain a formula that will work well for all values of r.

Formula for Ratios n up to 0.70: A good formula for values of n up to 0.70 is as follows:

$$k = \frac{64 - 3n^4}{64 - 16n^2} \tag{5}$$

This formula is evidently an approximate summation of the series in Formula (2), since when the numerator is divided by the denominator,

$$k = 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \frac{1}{256}n^6 + \frac{1}{1024}n^8 + \text{etc.} \tag{6}$$

The series in Formula (2) becomes, when reduced,

$$k = 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \frac{1}{256}n^6 + \frac{1}{656.36}n^8 + \text{etc.} \tag{7}$$

The first four terms of Formulas (6) and (7) agree; but, beginning with the fifth term, this agreement ceases, and the two formulas drift apart rapidly. The following formula is based on the values calculated for the table previously referred to, the formula having been derived by means of these values.

$$k = 1 + 0.002011n + 0.228989n^2 + 0.075834n^3 - 0.09974n^4 + 0.066146n^5 \tag{8}$$

This gives good results for all values of n, the greatest departure being between 0.9 and 1. Since Formula (5) will give five significant figures correct up to $n = 0.70$ (by Formula (5), $k = 1.126775 +$ for $n = 0.70$), and since it is easier to apply than Formula (8), it would be well to use Formula (5) for all values of n up to 0.70.

Formula for Ratios n between 0.70 *and* 1: For values of n between 0.70 and 1, Formula (9), which follows, has been derived.

$$k = 0.147202 + 5.355178n - 13.197449n^2 + 16.892262n^3 - 10.613854n^4 + 2.689901n^5 \tag{9}$$

This must not be used for values of n less than 0.70, for which $r = 5\frac{2}{3}$. Formula (9) will give at least five significant figures correct for all values of n between 0.70 and 1, or, in other words, for all values of r between $5\frac{2}{3}$ and ∞; it is therefore sufficiently accurate for all practical purposes. In the table, which may be used instead of the formula, the values of k have been limited to five significant figures, because if they were given to a greater number of figures, it would be necessary to use second and even third differences for the larger values of n, when interpolating for intermediate values of k.

Example: Find the periphery of an ellipse, the long and short diameters of which are 7 feet, 5 inches, and 9 inches, respectively. Since 7 feet, 5 inches = 89 inches, $n = \frac{89 - 9}{89 + 9} = 0.816326$. From the table, obtain by interpolation $k = 1.1723 + (1.1768 - 1.1723) \times 0.6326 = 1.1751$. The value $0.6326 = \frac{0.816326 - 0.81}{0.82 - 0.81}$. Hence, by Formula (1),

$$p = \frac{3.1416}{2}(89 + 9) \times 1.1751 = 180.89 \text{ in.} = 15 \text{ ft., } 0.89 \text{ in.}$$

Length of Chord and Height of Circular Segment. Two simple formulas for calculating the length of a chord and the height of circular segments are as follows (see Fig. 23):

$$H = R - \sqrt{R^2 - W^2}$$

$$W = \sqrt{R^2 - (R - H)^2}$$

where

W = one-half length of chord;
R = radius of circle;
H = height of segment.

The fact that W represents but one half the length of the chord simplifies the formulas greatly, but this fact should be borne in mind when using the formulas.

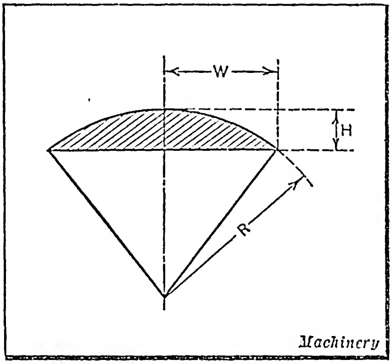

Fig. 23. Circular Segment Diagram

Formulas for Finding the Area of a Circular Segment. The accompanying table "Comparison of Various Segment Area Formulas" contains values which furnish a comparison between three approximate formulas for finding the areas of circular segments, and the exact formula. In the table, the four formulas are arranged and numbered according to the accuracy of the results obtainable by each. The exact formula will be given first.

$$A = \tfrac{1}{2}[rl - c(r - h)] \qquad (1)$$

in which c and h are known (see Fig. 24);

$$r = \frac{c^2 + 4h^2}{8h}; \qquad \text{and}$$

$$l = \frac{\pi r \alpha}{180} \text{ in which } \sin\frac{1}{2}\alpha = \frac{c}{2r}$$

Formula (1) will give exact results provided the value of l is very accurately calculated. The value of l is usually expressed as follows:

$$l = 0.01745r\alpha, \text{ in which } \alpha \text{ is expressed in degrees.}$$

A still closer approximation will be obtained by using the following constant:

$l = 0.00000484814 r\alpha$, in which α is expressed in seconds.

Formula (1) may also be written in the form:

$$A = r^2\left(\frac{\alpha}{2} - \sin\frac{\alpha}{2} \times \cos\frac{\alpha}{2}\right)$$

In this latter formula α is expressed in radians and not in degrees.

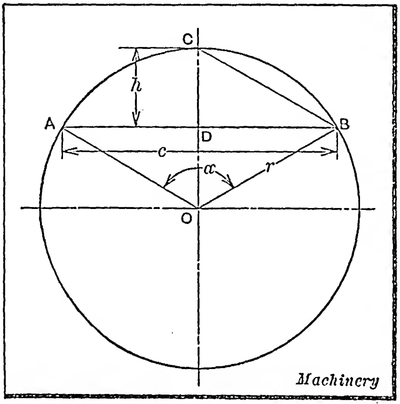

Fig. 24. Diagram used in deriving Formula for finding Area of Segment

$$A = \frac{4h^2}{3}\sqrt{\frac{c^2}{4h^2} + 0.392} \quad (2)$$

This formula gives results (for segments less than a semicircle) which will be correct to three figures.

$$A = \frac{h^3}{2c} + \frac{2ch}{3} \qquad (3)$$

This formula is much more easily handled than Formula (2), but Formula (2) is the best approximation. The derivation of Formula (3) may be followed by referring to the diagram, Fig. 24. The area of the segment may be written:

$$ACBA = \frac{\text{arc } ACB \times OB}{2} - \frac{AB \times OD}{2}$$

$$OB = \frac{c^2 + 4h^2}{8h}$$

and

$$OD = OB - h = \frac{c^2 - 4h^2}{8h}$$

The length of arc ACB, when expressed according to Huygens' approximation, is

$$ACB = \frac{8BC - AB}{3}$$

$$BC = \sqrt{\left(\frac{c}{2}\right)^2 + h^2} = \frac{1}{2}(c^2 + 4h^2)^{\frac{1}{2}} = \frac{1}{2}c\left(1 + \frac{4h^2}{c^2}\right)^{\frac{1}{2}}$$

Expanding the binomial in the parentheses of the last expression, by the well-known binomial theorem

$$BC = \frac{1}{2}c\left(1 + \frac{2h^2}{c^2} - \frac{2h^4}{c^4} \cdots\right)$$

$$= \frac{c}{2} + \frac{h^2}{c} - \frac{h^4}{c^3}, \text{ approximately}$$

Comparison of Various Segment Area Formulas

Values of c and h, Inches	Formula (1)	Formula (2)	Formula (3)	Formula (4)
$c = 10;\ h = 5$	39.2699	39.32770	39.58330	39.27000
Error..........	0.0000	− 0.05780	+ 0.31340	+ 0.00010
$c = 2;\ h = 1$	1.5708	1.57310	1.58330	1.57080
Error..........	0.0000	+ 0.00230	+ 0.01250	0.00000
$c = 4;\ h = 1$	2.7956	2.79430	2.79170	2.76660
Error..........	0.0000	− 0.00130	− 0.00390	− 0.02900
$c = 10;\ h = 3$	21.3736	21.36530	21.35000	21.16200
Error..........	0.0000	− 0.00830	− 0.02360	− 0.21160
$c = 10;\ h = 0.25$	1.6678	1.66748	1.66745	1.65178
Error..........	0.0000	− 0.00032	− 0.00035	− 0.01602
$c = 40;\ h = 1$	26.6859	26.67970	26.67910	26.42850
Error..........	0.0000	− 0.00620	− 0.00680	− 0.25740

The Huygens' formula for finding the length of ACB may now be written

$$ACB = \frac{1}{3}\left(4c + \frac{8h^2}{c} - \frac{8h^4}{c^3} - c\right)$$

$$= c + \frac{8h^2}{3c} - \frac{8h^4}{3c^3}$$

Substituting the values in the original equation

$$ACBA = \left(c + \frac{8h^2}{3c} - \frac{8h^4}{3c^3}\right) \times \left(\frac{c^2 + 4h^2}{16h}\right) - \frac{c}{2}\left(\frac{c^2 - 4h^2}{8h}\right)$$

Expanding and simplifying the terms in the above equation

$$ACBA = \frac{2ch}{3} + \frac{h^3}{2c} - \frac{2h^5}{3c^3}$$

We may reject the last term in this equation since the value of h is supposed to be small compared with that of c; therefore

$$\text{Area} = \frac{2ch}{3} + \frac{h^3}{2c} \text{ approximately}$$

This formula is accurate to within three figures when the height of the segment does not exceed the length of one fourth the chord; and the error will be less than 1 per cent when the segment is a semicircle.

$$A = \frac{h^3}{2c} + 0.6604ch \tag{4}$$

This formula gives results correct to four figures for a semicircle, but for general use is not so accurate as either of the other two approximate formulas. This fact will be apparent by inspecting the values given in the table.

To Determine Capacity of Cylindrical Tank. An approximate rule for finding the number of gallons that a cylindrical tank with flat ends will hold is as follows:

Double the diameter in inches, square the result, multiply by the length in feet, and divide by 100. If an accurate result is desired, add 2 per cent of the result obtained by the rule.

Let d = diameter in inches; l = length in inches; L = length in feet; and G = number of gallons. Then, $G = \dfrac{0.7854d^2l}{231} = 0.0034d^2l = 0.0034d^2 \times 12L = 0.0408d^2L = 0.04d^2L$ approximately $= \dfrac{(2d)^2L}{100}$.

Example: As an example, suppose that the tank is 60 inches in diameter and 18 feet long. Doubling the diameter gives 120; $120^2 = 14{,}400$; $14{,}400 \times 18 = 259{,}200$; dividing by 100 by pointing off two decimal places, the number of gallons is 2592, approximately. Two per cent of this is $2592 \times 0.02 = 51.84$, and $2592 + 51.84 = 2643.84$ gallons, the same result as would be obtained by substitution in the formula.

To Locate Cylindrical Tank Overflow at Given Capacity Level. A cylindrical tank 60 feet high and 12 feet in diameter is to be provided with an overflow opening at such a height in the head that the contents will overflow when the tank is two-thirds full. From Fig. 25, the problem may be stated mathematically as follows: Find a line AB so located that it divides a circle 12 feet in diameter into two parts, the area of one part being equal to two thirds of the area of the circle, and the area of the other part being equal to one third.

This problem cannot be solved by any direct mathematical formula. It is easily solved, however, by the use of a table of "Segments of Circles," such as is given in MACHINERY'S HANDBOOK. Tables of this kind are, in fact, prepared for the very purpose of solving problems like this, for which no simple mathematical formula can be provided. In the present case the problem resolves itself into finding the height *CD* of the segment of a circle the area of which is one third of the total area of the circle. The table referred to is based on a radius equal to 1. The area of a circle with a radius equal to 1 is 3.1416. One third of this area equals 1.0472. By referring to the table in the column "Area of Segment," it will be found that a segment having a center angle of 149 degrees and a height of 0.7328, very nearly meets the requirements. As the radius in the problem is 6 feet, the height *CD* would equal 6 × 0.7328, or 4.4 feet, very nearly.

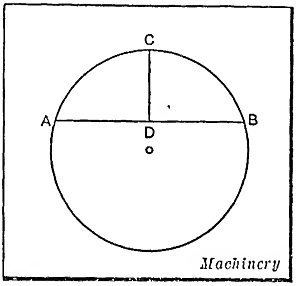

Fig. 25. Locating Cylindrical Tank Overflow at Given Capacity Level

The uses of the mathematical tables for solving many problems of this and similar kinds are perhaps not so well appreciated as they should be. To a man who must do a great deal of calculating, tables of "Powers, Roots and Reciprocals," "Circumferences and Areas of Circles," "Segments of Circles," "Lengths of Chords," etc., will save a great deal of time and facilitate the solution of problems and equations by trial and inspection to a great extent.

Capacity of Cylindrical Tank at Different Levels. Assume that a cylindrical tank is 90 inches in diameter, 18 feet long and so placed that its axis is horizontal. The problem is to calculate the contents in gallons for each six inches of depth. The ends of the tank are flat.

Solution: In Fig. 26 the horizontal lines represent the various levels, 6 inches apart. For any particular level, as *AB*, the contents in gallons is equal to the area of the segment *ACB* in square inches multiplied by the length of the tank in inches (to obtain the number of cubic inches), and the product divided by 231 which is the number of cubic inches in a gallon.

Let A = area of any segment;
h = height of segment;

r = radius; and
V = central angle.

Then,

$$A = \tfrac{1}{2}r^2 (V - \sin V)$$

V is measured in radians, and is found as follows: For any level, as AB,

$$\cos \frac{V}{2} = \frac{r - h}{r} = \frac{45 - 6}{45}$$

With the aid of a table of sines and cosines, find $\frac{V}{2}$; the result multiplied by 2 gives V. If a table giving angles in radians is not available, express V in degrees and decimal of a degree; then, in degrees, $V = 0.0174533V$. The area of the segment is then readily computed. By repeated applications of this process, the area of the segment that corresponds to each level is found. When the level FG is reached, the areas above it may be obtained by adding to the area last found the corresponding area below HI. Thus area $FGML$ = area $HIJK$, and area $HIJK$ = segment HCI − segment KCJ, both of which areas have been previously calculated.

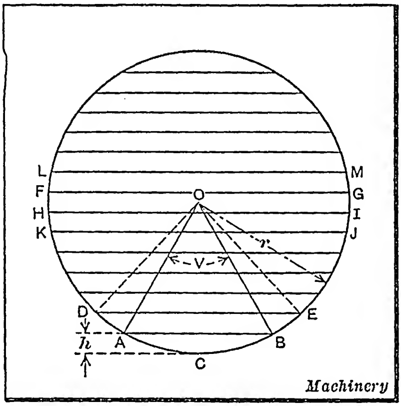

Fig. 26. To find Capacity of Cylindrical Tank at Different Levels

Since 18 feet = 216 inches, the capacity at any level is $C = \frac{216A}{231}$ $= 0.935065A$.

To Find Area of any Plane Figure. The two rules that are most frequently used for finding the area of a plane figure are the so-called trapezoidal rule and Simpson's rule; the latter is more accurate, but the former is a trifle easier of application. An actual example will make the process clear.

From Fig. 27, find the area included between the arcs ABC and AGC. First draw a line, either within or without the figure, but extending its entire length; in this case, the line AC will be the most

convenient. Divide this line into any even number of equal parts, say ten, and through the points of division, draw perpendiculars to AC; these perpendiculars are called ordinates. The ordinates are lettered h_0, h_1, $h_2 \ldots h_{10}$, the end ordinates, h_0 and h_{10}, being equal to 0 in this case. Measure the ordinates very carefully; add those which have odd subscripts (as h_1, h_3, etc.) and multiply the sum by 4; add those which have even subscripts (as h_2, h_4, etc., but not including the end ordinates) and multiply the sum by 2; add the two products and the two end ordinates, multiply the sum by the

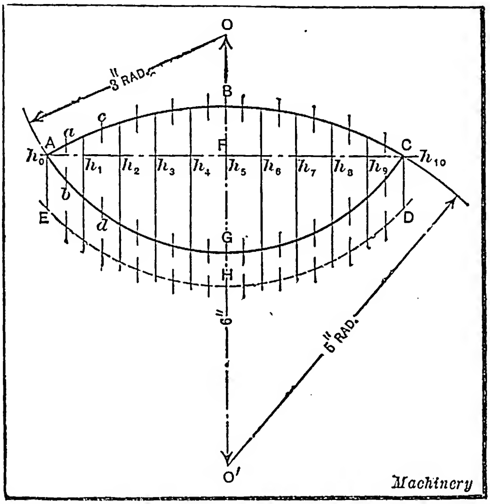

Fig. 27. Method of finding Area of any Plane Figure

distance between any two consecutive ordinates, and divide the product by 3; the result is the area by Simpson's rule. Expressed as a formula, the rule becomes

$$A = \frac{d}{3}\left[h_0 + h_{10} + 4\,(h_1+h_3+h_5+h_7+h_9) + 2\,(h_2+h_4+h_6+h_8)\right]$$

in which d = the distance between any two consecutive ordinates = $AC \div 10$, in this case. Assuming the values of this particular figure to be as given in the following example, then the area A = $(0.499 \div 3)\ [0 + 0 + 4\ (0.83 + 1.73 + 2 + 1.73 + 0.83) + 2\ (1.37 + 1.93 + 1.93 + 1.37)] = 6.93$ square inches.

By the trapezoidal rule, measure the ordinates half way between the equal division points (as ab, cd, etc.), find their sum, and multiply it by d, the distance between two consecutive ordinates. If the measurements are as follows then $A = 0.499\ (0.45 + 1.12 + 1.56 + 1.84 + 1.98 + 1.98 + 1.84 + 1.56 + 1.12 + 0.45) = 6.94$ square inches. By calculation, $A = 6.93$ square inches. Had the figure been $ABCDHE$, AE and CD being parallel to $O'O$, and

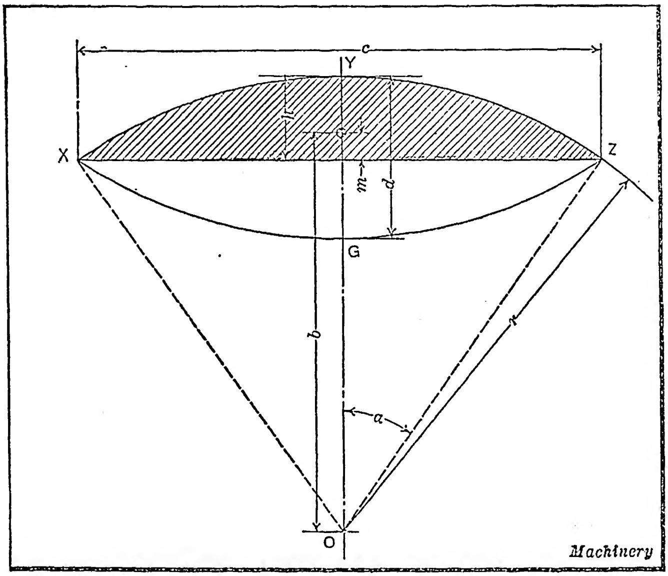

Fig. 28. Method of calculating the Volume of a Solid of Revolution

the radius $OH = 3.5$ inches, the area by Simpson's rule would be 9.79 square inches; by the trapezoidal rule, 9.84 square inches; and by calculation, 9.78 square inches. By the trapezoidal rule, AC may be divided into any number of equal parts, but Simpson's rule requires an even number.

Volume Solved by Pappus or Guldinus Rules. The problem is to find the volume of the solid shown in Fig. 28, any cross-section of which, perpendicular to the axis, is circular. Length $c = 18$ feet, and the middle diameter $d = 20$ inches. This solid is produced by the revolution of the segment of a circle XYZ about its chord XZ.

Solution: The volume of any solid of revolution may be found by means of the Pappus or Guldinus rules given in MACHINERY'S HANDBOOK.

Thus

$$V = 2\pi m A \tag{1}$$

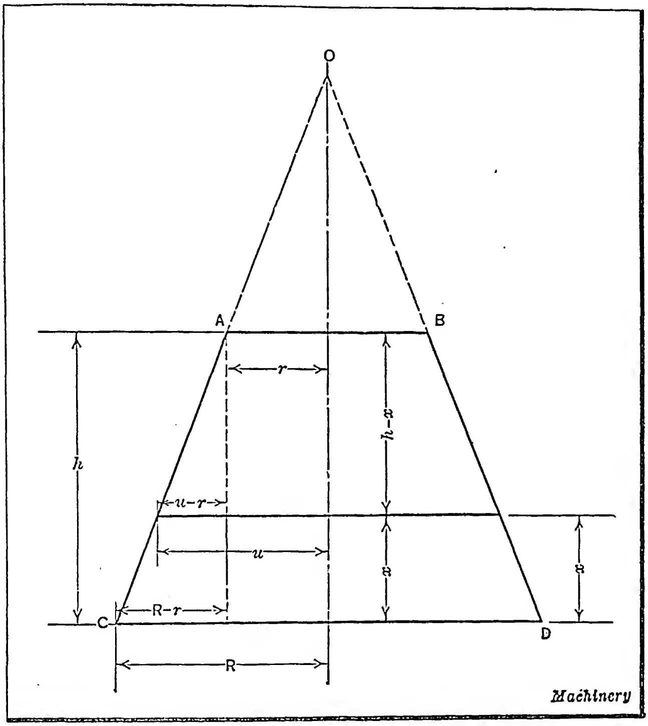

Fig. 29. To find Depth of Water in Conical Tank for Given Part of Total Capacity

in which V = volume of the figure;

A = area of segment XYZ; and

m = distance from the axis of revolution to the center of gravity of segment XYZ.

The area of a segment of a circle is found by the formula

$$A = r^2\,(\text{rad}\,\alpha - \sin\alpha \times \cos\alpha) \tag{2}$$

In the formula just given the expression "rad α" represents the angle in radians. The value of r is determined as follows:

$$r = \frac{c^2 + 4h^2}{8h} = \frac{46{,}656 + 400}{80} = 588.2 \text{ inches}$$

$$\text{Sin } \alpha = \frac{c}{2r} = \frac{216}{1176.4} = 0.183611$$

$$\alpha = 10 \text{ degrees } 34 \text{ minutes } 49 \text{ seconds}$$

$$\alpha \text{ (in radians)} = 0.1846586$$

$$\text{Cos } \alpha = \frac{r - h}{r} = \frac{588.2 - 10}{588.2} = 0.982999$$

Inserting the numerical values in Formula (2), we get

$$A = 1442.46 \text{ square inches}$$

The distance from the center O to the center of gravity of the segment is obtainable by the formula: $b = c^3 \div 12A$, from which we find $b = 582.20539$ inches.

Now

$$m = b - (r - h) = 4.00539 \text{ inches}$$

Inserting the numerical values for m and A in Formula (1), we find the volume of the solid to be 36,302 cubic inches or 21 cubic feet.

Problem Involving Volume of a Conic Frustum. The problem is to find the depth of water in a tank which is in the form of a frustum of a cone, when the tank is only partly filled. If a tank is 9 feet in diameter at the top, 10 feet in diameter at the bottom, and 5 feet high, what is the depth of water when it is one-quarter full?

Solution: In the diagram, Fig. 29, let $ABCD$ represent the vertical section of the given frustum taken through the center. Also, let r equal the radius of the top of the tank; R the radius of the bottom of the tank; h the height of the tank; n the number of parts in the total volume of the frustum, that is, the denominator of the fraction representing the part of the tank which is filled; and x the depth of water when the tank is $\frac{1}{n}$ full.

Denote the volumes of the three cones whose bases are circles having radii r, u, and R by the letters A, B, and C, respectively. Then,

$$A : C = r^3 : R^3 \qquad \text{and} \qquad B : C = u^3 : R^3$$

since corresponding volumes vary as the cubes of the radii or altitudes.

Therefore $\quad A = \frac{r^3}{R^3} C \quad$ and $\quad B = \frac{u^3}{R^3} C$

Volume of water: volume of frustum $= 1 : n$

Then $\quad \frac{\text{Volume of water}}{\text{Volume of frustum}} = \frac{C - B}{C - A} = \frac{C - (u^3 \div R^3)\, C}{C - (r^3 \div R^3)\, C}$

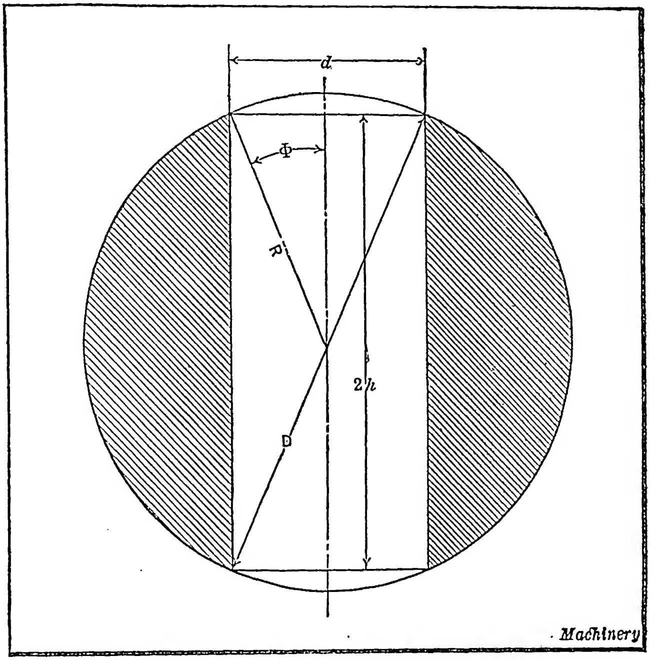

Fig. 30. To find Diameter of Hole equivalent to One Half the Contents of a Sphere

$$= \frac{R^3 - u^3}{R^3 - r^3} = \frac{1}{n}$$

from which

$$u^3 = \frac{R^3 (n - 1) + r^3}{n} \quad \text{and} \quad u = \sqrt[3]{\frac{R^3 (n - 1) + r^3}{n}} \qquad (1)$$

From the theorem of similar triangles,

$$\frac{u - r}{R - r} = \frac{h - x}{h} \quad \text{or} \quad u = R - \frac{x}{h}(R - r) \qquad (2)$$

Solving (1) and (2) for x, we find

$$x = \frac{h\left[R - \sqrt[3]{\frac{R^3 (n-1) + r^3}{n}}\right]}{R - r}$$

Reducing all dimensions to inches and putting $n = 4$, $R = 60$, $r = 54$, and $h = 60$ in this formula, we find $x = 13.97$. Hence, the depth of the water when the tank is one-fourth full is 13.97 inches.

To Remove One Half the Volume of a Sphere. The problem is to determine the diameter of the hole that must be drilled through the center of a 3-inch sphere to remove one half the contents. From the diagram (Fig. 30) it will be seen that the material to be removed by drilling consists of two equal spherical segments and a cylinder.

Solution: Let R = the radius of the sphere; and
$2h$ = the altitude of the cylinder.

Then

$R - h$ = the altitude of the spherical segment

and

$\sqrt{R^2 - h^2}$ = the radius of the base of the segment and the radius of the cylinder.

The volume of the two spherical segments in terms of R and h is:

$$2\left[\tfrac{1}{2}\pi (R - h)(R^2 - h^2) + \tfrac{1}{6}\pi (R - h)^3\right]$$

and the volume of the cylinder $= 2\pi h (R^2 - h^2)$

To remove $\frac{1}{n}$ of the sphere we have 2 times the volume of one segment plus the volume of the cylinder $= \frac{1}{n}$ the volume of the sphere.

Therefore,

$$\pi (R - h)(R^2 - h^2) + \frac{1}{3}\pi (R - h)^3 + 2\pi h (R^2 - h^2) = \frac{1}{n}\left(\frac{4}{3}\pi R^3\right)$$

Expanding and combining terms

$$\frac{4}{3}\pi (R^3 - h^3) = \frac{1}{n}\left(\frac{4}{3}\pi R^3\right)$$

Removing the factor $\frac{4}{3}\pi$,

$$R^3 - h^3 = \frac{1}{n} R^3$$

Again, referring to the diagram, $h = R\cos\phi$. Substituting this value for h in the equation just given, and removing the factor R^3,

$$\text{Cos}^3\,\phi = 1 - \frac{1}{n} = \frac{n-1}{n}$$

and

$$\text{Cos}\,\phi = \sqrt[3]{\frac{n-1}{n}}$$

To remove one half the volume of the sphere, let $n = 2$.

Then

$$\text{Cos}\,\phi = \frac{1}{\sqrt[3]{2}}$$

and

$$\phi = 37 \text{ degrees } 28 \text{ minutes}$$

Let D = diameter of sphere; and
d = diameter of hole.

Then,

$$d = D\sin\phi$$

For a 3-inch sphere this gives $d = 1.825$ inches.

To Find Radii and Number of Teeth of Gears when Center Distance and Ratio are Known. When the center distance and the speed ratio of two gears in mesh are known and the pitch radius of each gear is desired, divide the center distance by 1 plus the ratio of the pinion R.P.M. to the gear R.P.M. to obtain the pitch radius of the pinion. Then the difference between the center distance and the pitch radius of the pinion equals the pitch radius of the gear.

Example: Given the center distance between two gears, 12¼ inches; revolutions per minute of pinion, 80; revolutions per minute of gear, 60.

To find: Pitch radius of pinion and of gear

$$\text{Ratio of gears} = \frac{80 \text{ R.P.M.}}{60 \text{ R.P.M.}} = 1\ 1/3$$

$$\text{Pitch radius of pinion} = \frac{12\frac{1}{4} \text{ inches}}{1 + 1\ 1/3} = 5\frac{1}{4} \text{ inches}$$

$$\text{Pitch radius of gear} = 12\frac{1}{4} \text{ inches} - 5\frac{1}{4} \text{ inches} = 7 \text{ inches}$$

If the diametral pitch has been determined and the center distance between the gears is known, the number of teeth in the pinion can be found by taking twice the center distance

times the diametral pitch divided by 1 plus the ratio of the pinion R.P.M. to the gear R.P.M. The number of teeth in the gear can be found by subtracting the number of teeth in the pinion from twice the center distance multiplied by the diametral pitch.

Example: Given the center distances between two gears, $12\frac{1}{4}$ inches; revolutions per minute of pinion, 80; revolutions per minute of gear, 60; diametral pitch, 4.

To find: Number of teeth of pinion and number of teeth of gear

$$\text{No. of teeth in pinion} = \frac{2 \times 12\frac{1}{4} \times 4}{1 + 1\,1/3} = 42 \text{ teeth}$$

$$\text{No. of teeth in gear} = 2 \times 12\frac{1}{4} \times 4 - 42 \text{ teeth} = 56 \text{ teeth}$$

Note that in this last example if the center distance had been 12 inches instead of $12\frac{1}{4}$, the number of teeth in the pinion would have been 41 1/7. It will be seen from this that in order to retain a standard diametral pitch, some flexibility of center distance would be required.

Replacing Spur Gears with Helical Gears without Changing Center Distance. When spur gears on some existing machine are to be replaced either by single-helical or double-helical (herringbone) gears and both center distance and ratio must be retained, it may be possible to cut the helical or herringbone gear with a hob or cutter of standard pitch by making the helix angle special.

Rule: Select a hob or cutter having a diametral pitch equivalent to a slightly smaller tooth than that of the spur gearing to be replaced. Divide diametral pitch of spur gearing by diametral pitch of hob selected, thus obtaining cosine of helix angle required.

$$\begin{array}{c}\text{Cos. special}\\ \text{helix angle}\end{array} = \frac{\text{Diametral pitch of spur gear}}{\text{Diametral pitch of hob}}$$

Example 1: A machine has shafts connected by spur gears having 30 and 90 teeth, respectively, of 6 diametral pitch. Determine pitch of hob and helix angle of herringbone gears to replace spur gears. If a spur-gear hob of 7 diametral pitch is used,

$$\text{Cos special helix angle} = \frac{6}{7} = 0.85714$$

Hence helix angle equals 31 degrees, and with this angle the diametral pitch in the plane of rotation will be 6, like the spur gears which are to be replaced. If a hob of ½-inch circular pitch is used (½-inch circular pitch is equivalent to 6.2832 diametral pitch), then

$$\text{Cos special helix angle} = \frac{6}{6.2832} = 0.95493$$

Hence helix angle equals 17° 16′. If the herringbone gear teeth are cut to this angle with a hob of ½-inch circular pitch, the diametral pitch in plane of rotation will be 6; hence pitch diameter equals $N \div 6$. The tooth depth in this case would be based upon the normal pitch (½ inch circular or 6.2832 diametral).

Example 2: The spur gears, Example 1, are to be replaced by herringbone gears and a hob of 4 module (metric) is available. Determine helix angle.

The equivalent diametral pitch equals $25.4 \div 4 = 6.35$; hence

$$\text{Cos special helix angle} = \frac{6}{6.35} = 0.94488$$

The equivalent angle is 19° 7′ and the diametral pitch in the plane of rotation is 6, as required.

Width of Stock Required for Multiple Circular Blanking. Stamping circular blanks from strip steel is a very common requirement. For such work, the width of the strip for a given

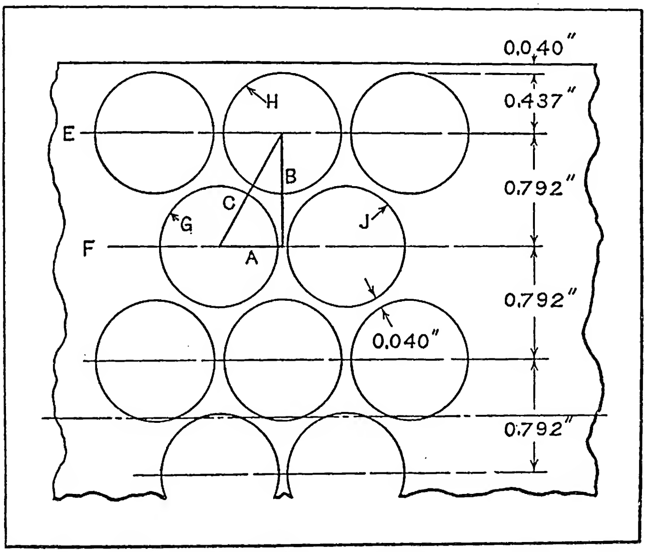

Fig. 31. Illustrating Method of Finding Width of Stock for Multiple Circular Blanking

number of rows of blanks can be calculated by common arithmetic. To illustrate the procedure, let it be assumed that six rows of circular blanks are to be punched across the strip, the diameter of each blank to be 0.875 inch. On a cold-rolled strip steel of No. 2 or No. 3 temper, ranging from 0.012 to 0.025 inch in thickness, a bridge 0.040 inch wide between the blanks and at the outer edges might be considered as suitable.

In Fig. 31, which shows one-half of the width of such a steel strip, the rows of blanks are laid out so that the center of a blank in one row is vertically above the center of the bridge between the two adjacent blanks in the adjoining row. In this diagram, the triangle ABC is drawn so that side B connects the center of one blank and the center of the bridge directly below it and represents the vertical distance between the center line of each row; side C connects the centers of the two adjacent blanks G and H; and side A connects the center of the lower blank with the center of the adjoining bridge.

It will be seen from this diagram that side A = the radius of the blank + ½ the width of the bridge between the blanks G and J. Thus, the value of $A = 0.437 + 0.020 = 0.457$ inch. The side C = the diameter of a blank + the entire bridge, or $0.875 + 0.040 = 0.915$ inch. The value of B is unknown.

Solving the right-angle triangle by the formula, $B = \sqrt{C^2 - A^2}$, we have $B = 0.793$ inch, which is the distance from the center of one row of blanks to the center of the next row across the strip. To find the width of the strip, multiply 0.793 by 5, and add the diameter of one blank and the width of the bridge on each side of the strip. Thus, $0.793 \times 5 + 0.875 + 0.080 = 4.920$ inches, the width of the strip required.

CHAPTER VIII

CONSTRUCTING ALIGNMENT CHARTS

LOGARITHMS are a valuable aid to engineers and draftsmen in facilitating lengthy calculations both by hand and by the slide-rule which has logarithmic scales. They are of further use in the construction of nomographic or alignment charts which make possible the rapid completion of a number of computations based on the same formula. Thus, a problem containing a number of factors can be drawn into chart form so that correct results may be obtained even though the mathematical processes are not fully understood.

In their simplest form, logarithmic alignment charts permit the rapid multiplication or division of two factors to get a third. Thus, in the equation $A \times B = C$, A, B and C are represented by vertical lines on which are laid out logarithmic scales with suitable values. By placing a straight-edge across the vertical line representing factor A and the vertical line representing factor B so that it coincides with the proper number indicated on the logarithmic scale of each, the answer or correct value of C may be read as a number at the point where the straight-edge crosses the logarithmic scale on vertical line C.

Laying Out a Chart. Probably the one thing that tends to discourage a more general use of these charts is the fact that they appear difficult to draw. Actually, the chart for simple multiplication or division just mentioned may be constructed quite easily. Thus, the scale representing factor A and the scale representing factor B may be any convenient length and distance apart. To lay out suitable logarithmic scales on each of the two vertical lines representing factors A and B, a piece of logarithmic cross-section paper is, in each case, placed at such an angle with one of the vertical lines, that the two extreme divisions desired just coincide with the horizontal extension of the ends of the line. The intermediate divisions can then be marked off by the intersection of parallel horizontal lines drawn from the intermediate divisions of the cross-section paper. The numbers or scale values can then be placed against each division. The principle involved will be illustrated later in connection with Figs. 2 and 3.

Next comes the problem of placing the vertical line to represent the factor C at the proper horizontal distance from lines A and B. This can be accomplished by drawing two lines, each connecting two numbers on the A and B scales, the products of which are equal, such as point 2 on line A and point 100 on line B and point 10 on line A and point 20 on line B. The point at which these two lines intersect marks one point through which the vertical line C must pass. This point will also be given the numerical value of the product of the two sets of numbers used on scales A and B which would be 200 in this case. The vertical line C is now drawn and several of the points on it can be numbered simply by placing a straight edge across lines A and B indicating the intersection on line C and placing a number on this point which equals the product of the corresponding numbers A and B. With these points established and numbered, the logarithmic scale can be laid out, again with the aid of a piece of logarithmic cross-section paper placed at a suitable angle with line C. The remaining numbers can now be placed against the intermediate divisions. The chart is then ready for use. When the ruler is laid across any two numbers on scales A and B, their product can be read directly on scale C. When the ruler is laid across any two values on the A and C scales, the quotient of C divided by A can be read directly on the B scale. With a little practice, anyone familiar with ordinary mathematics can easily learn how to chart his own formulas in a similar manner.

Charting a Formula with Four Variables. In order to illustrate in detail the steps required in charting an equation, the following formula containing four variables will be used as an example:

$$d = \sqrt[3]{\frac{321{,}000\ H.P.}{nS}} \tag{1}$$

This formula gives the diameter d of a solid shaft required to transmit a given horsepower *(H.P.)*, when revolving at n revolutions per minute. In the formula:

d = diameter of shaft, in inches;
$H.P.$ = horsepower to be transmitted;
n = number of revolutions per minute; and
S = allowable torsional shearing stress, in pounds per square inch.

Formula (1) can be written in the form:

$$H.P. = \frac{nSd^3}{321{,}000} \tag{2}$$

By taking logarithms of both members of (2):

$$\text{Log } H.P. = \text{Log } n + \log S + 3 \log d - \log 321{,}000 \qquad (2a)$$

Let it be required to chart Formula (2) for values of n ranging from 100 to 600 revolutions per minute; values of S ranging from 4000 to 10,000 pounds per square inch; and shafts ranging from 1 inch to 4 inches in diameter.

Then, for the range of values given, it will be noted that $\log n$ varies from $\log 100 = 2$, to $\log 600 = 2.77815$; or a logarithmic range of 0.77815; $\log S$ varies from $\log 4000 = 3.60206$, to $\log 10{,}000 = 4$; or a logarithmic range of 0.39794. Likewise, $\log d$ varies from $\log 1 = 0$, to $\log 4 = 0.60206$; or $\log d^3$ varies from $3 \times \log 1 = 0$, to $3 \times \log 4 = 1.80618$, giving a logarithmic range for d^3 equal to 1.80618.

Since only three variables can be charted in each step of the process, Formula (2) is transposed to this form:

$$Sd^3 = \frac{321{,}000 \, H.P.}{n} \qquad (3)$$

Taking logarithms of both members of (3), and assuming each member equal to $\log k$:

$$\text{Log } S + 3 \log d = \log k \qquad (4)$$

$$\text{Log } 321{,}000 + \log H.P. - \log n = \log k \qquad (5)$$

Equation (2) can then be charted in two steps of three variables each.

Equation (4) can be charted for the three variables S, d, and k, as indicated in Fig. 1, by placing the axis of k somewhere between the axes of n and d.

If it is desired to limit the graduated length of the axes to, say, 8 inches or less, a suitable scale value must be selected for each axis. Let these scale values be represented by l_1 for the S axis; l_2 for the d axis; and l_3 for the k axis.

To find approximate values for l_1, l_2, and l_3, divide the given logarithmic range as previously determined for each variable by 8; then,

For the axis of S, $l_1 = 0.39794 \div 8 =$ say, 0.05;

For the axis of d, $l_2 = 1.80618 \div 8 =$ say, 0.25.

Then:

$$l_3 = l_1 + l_2 = 0.05 + 0.25 = 0.30$$

Taking $l_1 = 0.05$, the measured length of the S axis from the graduation mark for $S = 4000$ to the mark for $S = 10{,}000$ is $0.39794 \div 0.05 = 7.96$ inches.

Taking $l_2 = 0.25$, the measured length of the d axis from the mark representing $d = 1$ to the mark for $d = 4$ is $1.80618 \div 0.25 = 7.225$ inches.

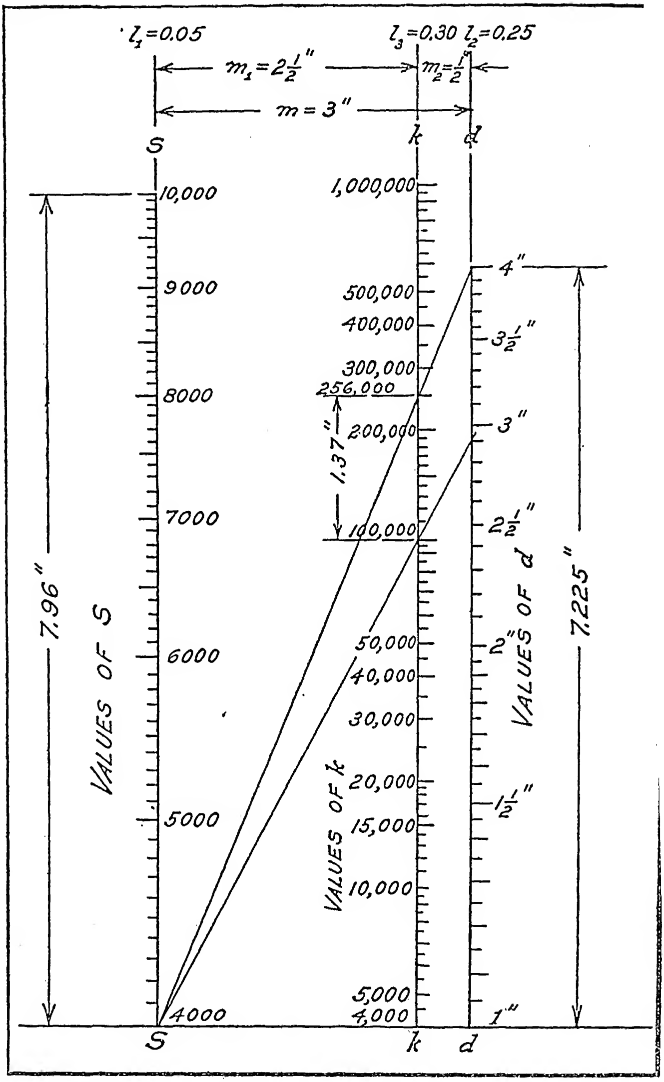

Fig. 1. The First Step in Constructing the Alignment Chart is to Lay Out the Axis Scales for Three of the Four Variables in the Formula

In the finished chart correlating the four variables n, S, d, and $H.P.$, the axis of k is used simply as a "supporting axis" and need not be graduated. For the purpose of illustrating a chart with only three variables, however, this axis will be graduated as explained below.

Locating and Graduating the Axes. Having decided upon the scale values l_1 and l_2 for the axes of S and d, respectively, these axes can now be located with reference to the third axis k. Since the method outlined earlier in the chapter for locating the third axis of an alignment chart is not easily applicable in this case, a different method will be used. Referring to Fig. 1, for the correct location of the third axis k, the ratio $m_1 \div m_2$ must equal the ratio $l_2 \div l_1$ or:

$$\frac{m_1}{m_2} = \frac{l_2}{l_1} = \frac{0.25}{0.05} = 5$$

Therefore, since $m_1 = 5 \times m_2$, the d axis can be placed, say, ½ inch to the right of the k axis ($m_2 = \frac{1}{2}$); and the S axis $5 \times \frac{1}{2} = 2\frac{1}{2}$ inches to the left of the k axis ($m_1 = 2\frac{1}{2}$).

The next step is to graduate the axes, which may be done most simply by using a suitable logarithmic scale, such as the scale on a slide-rule, or by using logarithmic paper, which is available in almost any stationery store handling draftsmen's supplies. This paper can be obtained graduated from 1 to 10 in a length of 5 inches, and also in a length of 10 inches.

The S axis is to be graduated from 4000 to 10,000 in a length of 7.96 inches. Using a sheet of logarithmic paper ruled from 1 to 10 in a length of 10 inches, the S scale can be graduated by the method shown in Fig. 2. The graduations of the logarithmic paper are shown on the line CD, the graduated length from 4 to 10 being 3.95 inches, approximately. Draw the line AB in such a manner that the length from a to g is 7.96 inches. This line cuts the ruled lines on the logarithmic paper from 4 to 10 at the points a, b, c, d, e, f, and g; the point a coincides with 4 on the logarithmic scale; b with 5 on the logarithmic scale; c, with 6, etc.

Point a on the derived scale is marked 4000; b, 5000; c, 6000; etc. Intermediate graduations such as those between a and b, are put in on the derived scale in a similar manner, as shown. The graduations representing the values of S obtained in this manner from the logarithmic paper can then be transferred directly to the S axis of Fig. 1.

The axis d can then be graduated in a similar manner for values of d ranging from 1 to 4, as shown. To make this scale

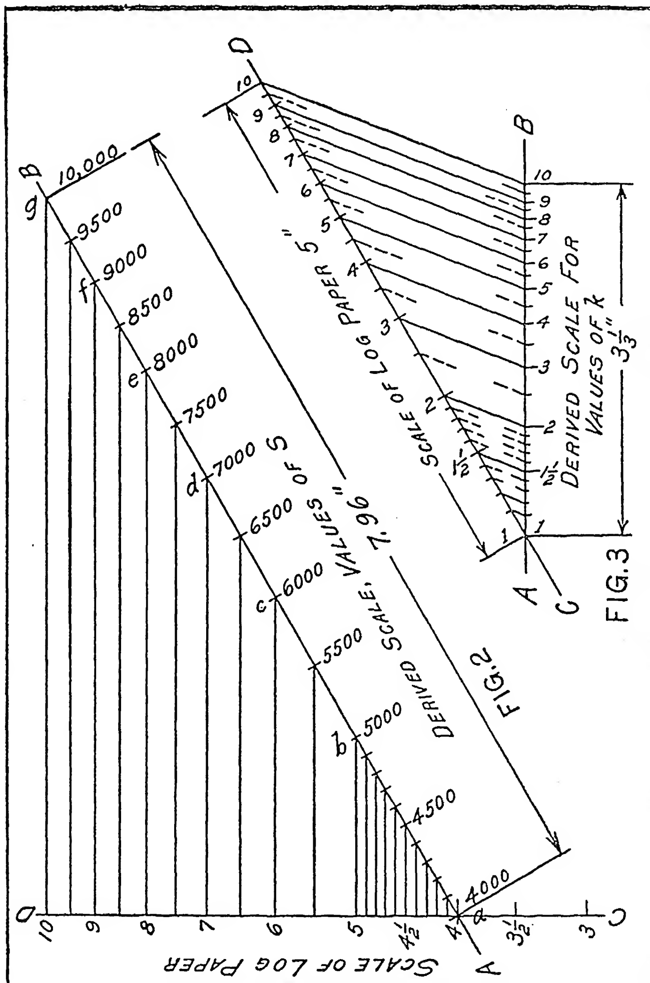

Fig. 2. Showing Method for Extending a Logarithmic Paper Scale to Construct a Longer Chart Scale
Fig. 3. Showing Similar Method of Constructing a Shorter Chart Scale from Logarithmic Paper Scale

in a length of 7.225 inches, point a on the derived scale, Fig. 2, is connected with 1 on the logarithmic paper, and point g with the ruled line representing the value 4; the measured length from a to g in this case being 7.225 inches. Care should be taken to mark the graduations as accurately as possible, using a sharply pointed pencil for the purpose.

Assuming that the k axis, Fig. 1, is to be graduated also (not required in the completed chart), a starting point may be found by connecting 4000 on the S axis with 4 inches on the d axis. A straight line connecting these two points cuts the k axis in graduation $k = 256{,}000$, determined as follows: By Equation (4),

$$\text{Log } S + 3 \log d = \text{Log } k \quad \text{or}$$

$$\text{Log } 4000 + 3 \log 4 = 3.60206 + 1.80618 = 5.40824 = \log k;$$

$$\text{hence, } k = 256{,}000$$

It is simpler, however, to start the graduation of the k axis from the marking for $k = 100{,}000$. This mark will be located 1.36 inches below the mark for $k = 256{,}000$; that is,

$$\frac{5.40824 - 5}{0.30} = 1.36$$

in which 5.40824 is the logarithm of 256,000; 5, the logarithm of 100,000; and 0.30, the scale value l_3 for the k axis, as given in the foregoing.

When $S = 4000$ and $k = 100{,}000$, then

$$3 \log d = \log k - \log S = 5 - 3.60206 = 1.39794$$

$$\text{Log } d = 1/3\ (1.39794) = 0.46598, \text{ and } d = 2.924 \text{ inches}$$

A straight line connecting 4000 on the S axis with 100,000 on the k axis will, if extended to the d axis, cut that axis at graduation $d = 2.924$. It should be noted that the graduations on the d axis are actually those representing values of 3 log d (or d^3), but that they are read off as values d and not d^3.

The k axis can now be graduated, starting at graduation 100,000, which is 1.36 inches below the marking for $k =$ 256,000. A scale for the k axis may be prepared as shown in Fig. 2 and already explained for the S axis. In this case, however, note that when $S = 4000$ and $d = 1$, the least value of k is $k = S \times d^3 = 4000 \times 1 = 4000$, or $\log k = \log 4000 + 3 \log 1 = 3.60206 + 0 = 3.60206$, and $k = 4000$.

Similarly, when $S = 10{,}000$ and $d = 4$, the greatest value of k required is $k = 10{,}000 \times 64 = 640{,}000$, or $\log k = \log 10{,}000 + 3 \log 4 = 4 + 1.80618 = 5.80618$, and $k = 640{,}000$.

It is easy to graduate the k axis by preparing a logarithmic scale ranging from 1 to 10 in a length of:

$$\frac{\log 10 - \log 1}{l_3} = \frac{1 - 0}{0.30} = 3\ 1/3 \text{ inches}$$

The k axis may then be graduated from the 100,000 mark already located (Fig. 1), both upward and downward as shown, using the derived scale and repeating as required. The method of preparing this reduced logarithmic scale from 1 to 10 in 3 1/3 inches is shown in Fig. 3, where *CD* is the original scale on the logarithmic paper and *AB* is a line drawn through graduation 1 on that scale at any suitable angle. Measure off a length of 3 1/3 inches on *AB;* connect 10 on *CD* with 10 on *AB* and draw the other lines representing the intermediate graduations parallel to this line.

The chart Fig. 1 correlates the three variable quantities S, d, and k according to the equation $S \times d^3 = k$ or $\log S + 3 \log d = \log k$.

Given any pair of values of S and d, the corresponding value of k may be found by connecting the given value of S with the given value of d; where the line intersects the k axis, read off the corresponding value of k. If S and k are the given values, then connect S and k; where the line extended cuts the d axis, read off the corresponding value of d. In the same way, if d and k are given, connect the d and k values at the proper graduations, extend the line to the S axis, and read off the value of S.

Completing the Chart. Fig. 1 represents the first step of the process required to chart Equation (2) or (2a), except that the k axis need not be graduated in the completed chart, since it acts simply as a "supporting" axis for the others. This first step of the process consists in locating the S, k, and d axes with reference to one another, and graduating the S and d axes as shown. In the second step, the n and *H.P.* axes are located and graduated with reference to the k axis so as to satisfy the equation

$$\frac{321{,}000\ H.P.}{n} = k \qquad \text{or}$$

$$\text{Log } 321{,}000 + \log H.P. - \log n = \log k \qquad (5)$$

This last equation can be charted, according to the methods shown in Fig. 4, by placing the *H.P.* axis to the right of the k axis and the n axis to the left of the k axis. Since it is desired to limit the length of the axes to 8 inches or less, the proper values l_4 for the n axis and l_5 for the *H.P.* axis must be selected.

Since the values of n are to vary from 100 to 600, or log n

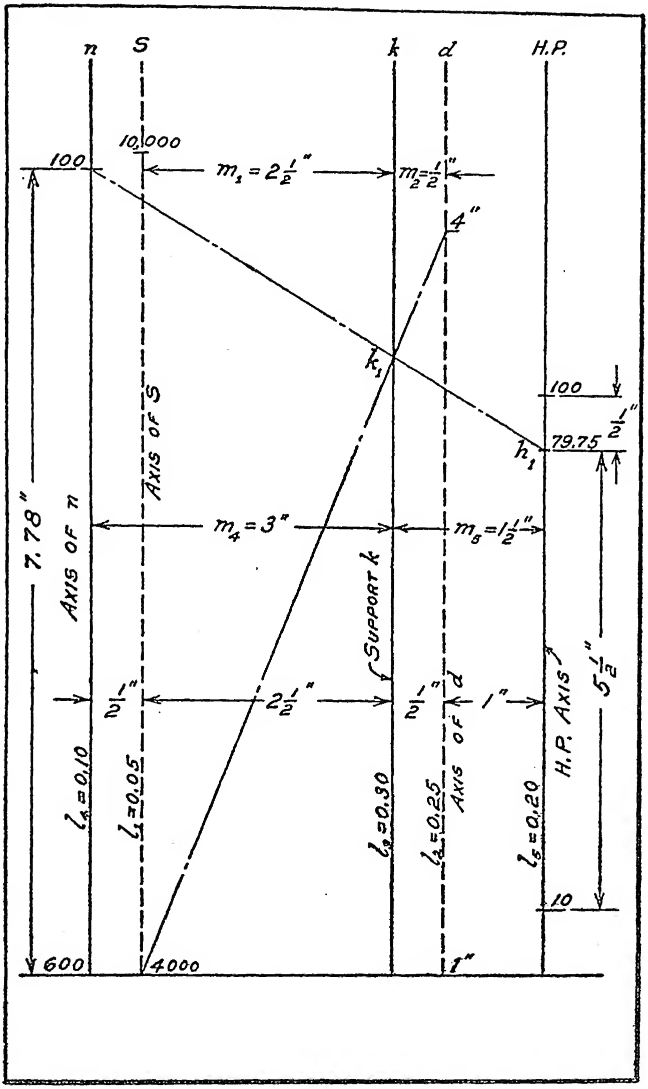

Fig. 4. The Final Step in Constructing the Alignment Chart is to Lay Out the Scale for the Fourth Variable—in this Case on the Horsepower Axis

from 2 to 2.77815, the logarithmic range is 0.77815; then 0.77815 ÷ 8 = 0.097. Hence, if l_4 is made equal to 0.1, the graduated length of the n axis from graduation 100 to graduation 600 will be 0.77815 ÷ 0.1 = 7.78 inches. The n axis must then be graduated from $n = 100$ to $n = 600$ in a length of 7.78 inches, but since log n is preceded by a minus sign, the n axis must be graduated downward instead of upward, beginning with 100 at the top as shown.

The *H.P.* axis is m_5 units to the right of the k axis, and the n axis, m_4 units to the left of the k axis. Now $l_4 + l_5 = l_3$, hence $l_5 = l_3 - l_4 = 0.3 - 0.1 = 0.2$, where l_3 is the scale value for the axis of k; l_4, the scale value for the axis of n; and l_5, the scale value for the axis of *H.P.* Then:

$$\frac{m_4}{m_5} = \frac{l_5}{l_4} = \frac{0.2}{0.1} = 2; \text{ and } m_4 = 2m_5$$

Hence, if m_5 is made equal to 1½ inches, m_4 equals 3 inches.

The d axis is now located ½ inch to the right of the k axis; the *H.P.* axis 1 inch to the right of the d axis, or 1½ inches to the right of the k axis; the S axis 2½ inches to the left of the k axis; and the n axis ½ inch to the left of the S axis, or 3 inches to the left of the k axis, and graduated in the opposite direction to the other axes. The k axis is not graduated. The finished chart will appear as in Fig. 5. The *H.P.* axis is still to be graduated as shown in that figure.

Graduating the Axis for the Horsepower. To find a starting point for the graduation of the *H.P.* axis, 4000 on the S axis is connected with 4 on the d axis, as shown in Fig. 4, and a point k_1 is marked on the k axis; then 100 on the n axis is connected with k_1, the line is extended to the *H.P.* axis, and the point h_1 marked on that axis. The value of *H.P.* at point h_1 is obtained from Equation (2a):

$$\begin{aligned}\text{Log } H.P. &= \text{Log } n + \log S + 3 \log d - \log 321{,}000 \\ &= \text{Log } 100 + \log 4000 + 3 \log 4 - \log 321{,}000 \\ &= 1.90173. \text{ Hence } H.P. = 79.75 \text{ at } h_1.\end{aligned}$$

It is much simpler, however, to start the graduation of the *H.P.* axis from the mark for *H.P.* = 100, or log 100 = 2. This mark for 100 H.P. is located a distance 0.491, or approximately ½ inch above the point h_1 for 79.75 H.P. This is determined as follows:

$$\frac{\log 100 - \log 79.75}{l_5} = \frac{0.09827}{0.2} = 0.491$$

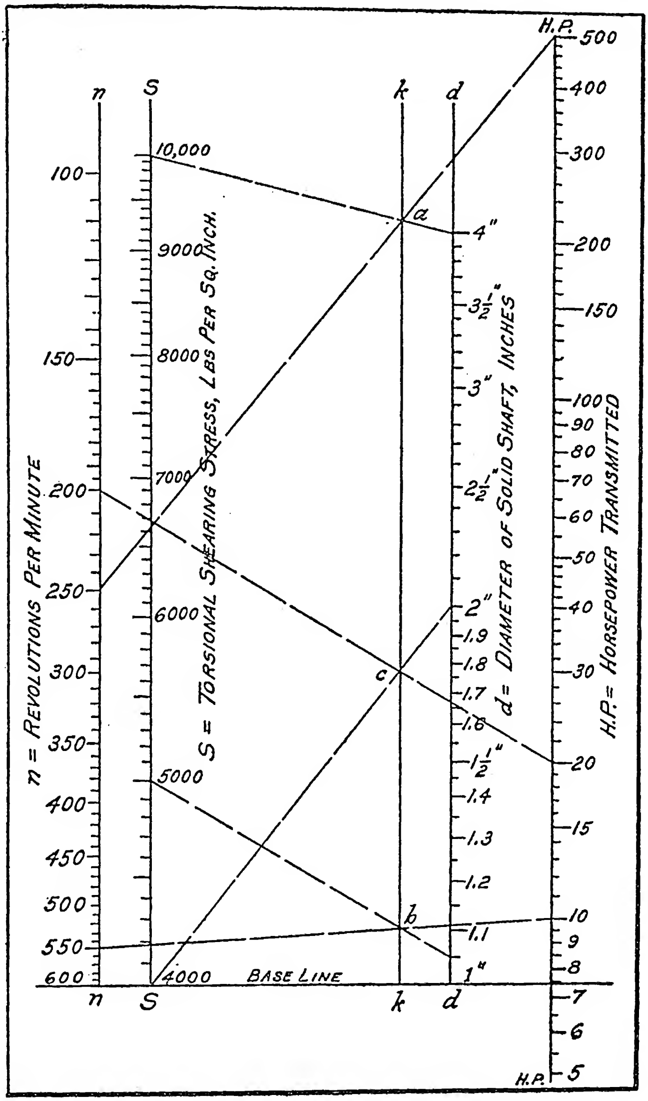

Fig. 5. Completed Alignment Chart for Solving Equation with Four Variables. Broken Lines Indicate Method of Using

The *H.P.* axis is graduated by a logarithmic scale reading from 1 to 10, or from 10 to 100 in a length of 5 inches, since

$$\frac{\log 10 - \log 1}{0.2} = \frac{\log 100 - \log 10}{0.2} = 5$$

This logarithmic scale can then be used to graduate the *H.P.* axis above and below the mark for 100 H.P. by putting in the proper values for the graduations thus located, as shown in Fig. 5. The values given on the *H.P.* axis are from 5 to 500, inclusive, but if the full range of the chart is to be utilized, the graduations can be extended as desired.

The following examples will illustrate the method of reading the chart and will serve as a check on the accuracy of its construction.

Example 1: Given the values $S = 10{,}000$; $d = 4$; and $n = 250$; find the horsepower transmitted.

Connect 10,000 on S, Fig. 5, with 4 on d, the line cutting the k axis at point a; connect 250 on n with point a on k, extend to the *H.P.* axis, and read off 500 horsepower, approximately.

To check this by Equation (2a):

$$\begin{aligned}\text{Log } H.P. &= \text{Log } 250 + \log 10{,}000 + 3 \log 4 - \log 321{,}000 \\ &= 2.39794 + 4 + 1.80618 - 5.50651 = 2.69761\end{aligned}$$

Hence, $H.P. = 498.4$, approximately.

Example 2: Given $H.P. = 10$; $S = 5000$; $n = 550$; find the diameter of the shaft d.

Connect 10 *H.P.* with 550 on n, marking point b on k axis; connect 5000 on S with the point b on k; extend to axis d and read off 1.05 inch, as the diameter of the shaft.

To check this, we have, by transposition of Equation (2a),

$$\begin{aligned}\text{Log } d &= 1/3\ (\log H.P. + \log 321{,}000 - \log n - \log S) \\ &= 1/3\ (\log 10 + 5.50651 - \log 550 - \log 5000) \\ &= 1/3\ (0.06718) = 0.02239\end{aligned}$$

Hence, $d = 1.05$ inch diameter

Example 3: Given $H.P. = 20$; $n = 200$; $d = 2$; find S in pounds per square inch.

Connect 20 *H.P.* with 200 on n, marking point c on the k axis; connect 2 on d with point c on k, extend to the S axis, and read off $S = 4000$, approximately.

From Equation (2a) we have:

$$\begin{aligned}\text{Log } S &= \text{Log } H.P. + \log 321{,}000 - \log n - 3 \log d \\ &= \log 20 + 5.50651 - \log 200 - 3 \log 2 \\ &= 3.60342\end{aligned}$$

Hence, $S = 4012$

In making an alignment chart similar to that shown for a given formula, it should be checked thoroughly with several

examples before being used. This method of checking will show any errors in the construction.

Any equation such as

$$x^{0.9} = u^{-1.6} \times v^{2.1} \times 0.6w^{1.5}$$

which can be written in the form

$$0.9 \log x = 2.1 \log v + \log 0.6 + 1.5 \log w - 1.6 \log u$$

can be charted with the axes parallel as in Fig. 5. The last equation can be written

$$0.9 \log x + 1.6 \log u = \log k$$

$$2.1 \log v + \log 0.6 + 1.5 \log w = \log k$$

Any other transposition in the original equation may be made as desired. The equation is then charted in two steps of three variables each.

If the transposition in the original equation is made as follows:

$$\frac{x^{0.9}}{v^{2.1}} = \frac{0.6w^{1.5}}{u^{1.6}} = k$$

or

$$0.9 \log x - 2.1 \log v = \log 0.6 + 1.5 \log w - 1.6 \log u = \log k$$

the v and u axes will be graduated in the opposite direction to the other axes, similar to the n axis of Fig. 5.

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